Embedding of Space-Time

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Abstract

Actual physical theories contain mysterious terms, such as Black Matter and Black Energy. This article introduces a geometrical approach, which offers new possible explanations for these topics.

Motivated by the merging of classical de-Sitter space (**dS**) and Anti-de-Sitter space (**AdS**) in a 6-D light-cone and the embedding of the Schwarzschild space (**Sch**) in a 6-D space, an embedding of stationary isotropic metrics into a common sphere in flat $R^{2,8}$ [1] is proposed. Integrability conditions for the embedding lead to inequalities between radial parameter and Schwarzschild radius, depending on the curvature radius of the global sphere. These inequalities define a minimal length structure in 3-D space.

Secondly, properties of the complex space, arising from an orthogonal projection of the Kaluza-Klein dimension onto the light cone (the Hopf fibration $S^1 \hookrightarrow S^9_2 \overset{p}{\to} CP^{1,3}$) are pointed out. In these coordinates any one of the embedded spaces become, in a coherent limit, minimal Lagrangian submanifolds, which are just 2-D spheres.

In section IV the ansatz is discussed under the point of the view of classical Kaluza-Klein theory [WL,Le,Str,Bl]. Motivated by this, a slight modification of the metric with astonishing features, including a negative vacuum energy, additional attractive forces on circular orbits with a modified gravitation constant and regions with euclidean metric, is discussed in section V . In section VI we extend this modification to the non-stationary case, which leads essentially to a time and space gravitation constant. We derive and discuss the related Hamilton-Jakobi equations and offer a simple particle model of the equations. In contrast to usual particles, this particle have a negative energy and it's anti-particle counterpart a positive one. Finally we point out, that the solutions offer explanations for the dark-matter and dark-energy effect.
The last section summarizes all and offers an idea, how perhaps one could explain, that it seems that particles could travel faster than light (if there will be a need for this) and some ideas concerning quantum effects.

I Introduction

Classically, **dS** and **AdS** metrics belong to the Friedman-Lemaitre cosmological model. **DS** describes an absolute homogenous, expanding, and **AdS** a contracting universe [St,Dr,Ri]. Due to the observations of an expanding universe in the last century, **AdS** is rejected as a model of space-time, but **AdS** geometry plays an important role in modern physical theories [RO]. The classical **dS** and **AdS** are just hyper-spheres in a five dimensional flat space, but the signature of both flat metrics differ (see [St, Mos] and Appendix **A**). While the causal structure of **dS** is causal, the one of **AdS** is acausal and allows closed time-like paths [Mos].

The two (to my knowledge) known embeddings of **Sch** into a flat 6-dimensional space, the Kasner and the Fronsdal embedding [Mo], have similar but different signatures. Note that Kasner's flat metric is of **AdS** type, while Fronsdal's is of **dS** type. Fronsdal's

1 I use $K^{n,m}$ for an n+m dimensional space and K^n_m for an n-dimensional space, both with m time-like dimensions.

embedding could be extended to the Kruskal metric [Mo, DP] and carries the same causal structure as **dS.** Kasner's embedding is acausal and not extendable. So Kasner's embedding is nowadays almost unvalued, but it will become the starting point of our common embedding, due to the observations below about the **dS/AdS** spaces. In this article I propose an embedding of **Sch** (together with a large set of isotropic, static metrics) into a manifold **M**, where the embedding combines Kasner's ansatz with the causality of the de-Sitter space and **M** contains also **dS** and **AdS** as submanifolds.
The first aim is now to merge the embeddings of **AdS and dS** and to look for similarities to the embeddings of the Schwarzschild space.

Some Notations are in order:

g, ds^2 is used for a metric (line element)
G for the Einstein gravitation tensor, without any cosmological constant.

The expression "Newton potential U_N of a metric", should be understood in the following way: Writing the form factor of the time coordinate of a metric as $g_{00} := g_{tt} = 1 + U$ one gets in the non-relativistic limit, via geodesic equation, the Newton potential U_N as

$$\ddot{r} \approx -\Gamma^r_{tt} = \frac{1}{2} \cdot g^{rr} \cdot g_{tt,r} \approx -\frac{1}{2} \cdot g_{tt,r} \approx -U_{,r}/2 \Rightarrow U_N \approx U/2 .$$

If ($(x_i, i=0 \ldots n)$ are coordinates in $\boldsymbol{R}^{d,(n-d)}$, the first three spatial entries are used as the usual 3-d spatial coordinates: $\boldsymbol{r} = (x_d, x_{d+1}, x_{d+2}),\ r := |(\boldsymbol{r})|$. The scalar product is given by

$$\langle x, y \rangle = \sum_0^{d-1} x_i \cdot y_i - \sum_d^{n-1} x_i \cdot y_i$$

dS and AdS as conic sections and the Schwarzschild metric

(For definition and a set of elementary properties of classical **dS**/**AdS**, see Appendix A).
The embedding of classical **AdS** and **dS** could be merged in the 6-dimensional flat space $\boldsymbol{R}^{2,4}$ as simple different sections of the same hyper sphere, the "light cone" **K**

$$\mathbf{K} = \{ x \in \boldsymbol{R}^{2,4} : x^2 := \langle x, x \rangle = 0 \}$$

One gets **dS** and **AdS** as the sections $x_1 = 1$ and $x_2 = 1$ of K. (We use here the indices 3,4,5 for the usual space dimensions.) The subspaces could be transformed into each other via a simple rotation of the $x_1 \text{ and } x_2$ axis around one of the others (e.g. the x_0 -axis). The radial part of the stationary metrics of **dS** and **AdS** reflects this also: For this consider the cone $x^2 + r^2 = y^2$.

dS: $$y = \alpha : \Rightarrow x^2 + r^2 = \alpha^2 \Rightarrow dr^2 + dx^2 = \frac{dr^2}{(1-(r/\alpha)^2)}$$

AdS: $$x = \alpha : \Rightarrow \alpha^2 + r^2 = y^2 \Rightarrow dr^2 - dy^2 = \frac{dr^2}{(1+(r/\alpha)^2)}$$

As is pointed out in Appendix B, using equal parametrization of **K** leads to qualitatively completely different kinds of metrics for **dS** and **AdS.** Circular coordinates induce a Friedman-Lemaitre-Robertson-Walker (FLRW) metric on **dS,** but a stationary metric on

AdS. Hyperbolic coordinates, on the other hand, induce the opposite kind of metrics, a stationary on **dS** and a FLRW metric on **AdS**. So it is senseless hoping to find a common parametrization of the hypersphere **M,** which is what we are looking for, which leads also to comparable types of metrics for **dS, AdS** and **Sch.**
As already mentioned, for **Sch** there exists also an embedding into, $R^{2,4}$ the Kasner embedding, while the embedding space of Fronsdal is $R^{1,5}$ [Mo]. The geometry of both embeddings is difficult and without any further similarities to **dS** and **AdS,** in particular it is not a submanifold of **K** and is not projective, while **dS** and **AdS** are projective spaces**.**

But there is another similarity between the Schwarzschild metric and the stationary metrics of **dS** and **AdS.**The radial part of the Schwarzschild metric $-g_{rr}=(1-r_0/r)^{-1}$ comes from a conic section:

$$x^2=4\mathrm{r}_0\cdot(r-r_0) \Rightarrow dr^2+dy^2=\frac{dr^2}{(1-(r_0/r))}$$

This conic section arises from $x^2+r^2=y^2$ through a rotation around the x-axis with angle $\pi/4$, followed by a translation $r\rightarrow r-r_0$ and finally cut at $y=2\cdot r_0$

$$y\rightarrow(y-r)/\sqrt{(2)}, r\rightarrow(r+y)/\sqrt{(2)} \quad \Rightarrow \quad x^2=(y-r)(y+r)\rightarrow x^2=2\,y\cdot r\rightarrow x^2=4\mathrm{r}_0(r-r_0)$$

Comparing the parameters of the cutting planes $y=2\cdot r_0$ and $y=\alpha$ leads to

$$\boldsymbol{r_0=\alpha/2} \tag{I.1}$$

A relation of this kind, between r_0 and α , we will get again and it will become an essentially part of the conclusions and interpretations in this paper.

II Embedding in a common Space

Due to the considerations in the last section it seems appropriate to restrict **M** to be a hypersphere in $\boldsymbol{R}^{2,k}, k>4$. Using k=5, i.e. seven dimensions, does not lead to essentially better results, moreover it is hard to understand why two independent extra space-like dimensions and only one time-like dimension should exist. For several reasons I also think that the number of dimensions must be even such that a complex structure could be defined on it. For k=6, i.e. in eight dimensions, one has exactly two copies of space-time. Now we got a redundant system. To shrink again the set of free parameters, we my assume some additional symmetries. Because of space isotropy, I would expect that the angles between the two sets of space dimensions should match. But then one loses again two degrees of freedom, i.e. two dimensions, and we would have just the same as for 6 dimensions. Without any symmetry demands it would be possible to embed all three spaces (**dS, AdS** and **Sch)** in flat $\boldsymbol{R}^{2,6}$. But than we also lose the symmetries found above for **dS/AdS** in 6 dimensions, and the embedding would carry the acausal structure of Kasner embedding.

So we end up with the 10 dimensional space $\boldsymbol{R}^{2,8}$. The number of dimensions is now two times the dimensions of the classical Kaluza-Klein theory. Also space isotropy and the de-Sitter space in mind, we start with the decomposition

$$\boldsymbol{R}^{2,8}=\boldsymbol{R}^{1,4}\circ\boldsymbol{R}^{1,4}$$

and the ansatz,

$$M=\{(x,y): x\in R^{1,4}, y\in R^{1,4} \text{ and } \langle x,x\rangle+\langle y,y\rangle=-\alpha^2\}.$$

In the following we use normalized coordinates, that is a normalized curvature radius $\alpha=1$. In a stationary, isotropic metric it means that time and radial parameters are measured in multiples of α. Kaluza-Klein theory postulates a S^1 submanifold so we write

$$\rho^2:=x_4^2+y_4^2\ ,\ r^2:=r_x^2+r_y^2\ ,\ r_x^2:=\sum_{j=1}^{3} x_i^2,\ \ r_y^2:=\sum_{j=1}^{3} y_i^2$$
$$\Rightarrow\langle x,x\rangle+\langle y,y\rangle=x_0^2+y_0^2-\rho^2-r^2=-1.$$

Again the three indices 1,2,3 are used as the usual space indices. The restriction that all angles between corresponding x- and y-space dimensions should be equal means:

$$x_i=r_i\cos(\phi),\, y_i=r_i\sin(\phi)\ \text{ for } i=1,2,3$$

Using spherical coordinates for the space coordinates and writing for the Kaluza-Klein S^1 - part $x_4=\rho\cos(\beta),\, y_4=\rho\sin(\beta)$, the metric on $\boldsymbol{R}^{2,8}$ becomes

$$ds^2=dx_0^2+dy_0^2-d\rho^2-\rho^2 d\beta^2-dr^2-r^2 d\phi^2-r^2 d\Omega \tag{II.1}$$

One way to recover **dS** and **AdS** is setting $\phi=constant$, $\beta=constant$ and

AdS: $\rho^2=2,\ x_0=A\cos(t),\, y_0=A\sin(t),\ \ A^2=1+r^2$

dS: $\rho=A\cosh(t),\ x_0=A\sinh(t),\ y_0=0,\ \ A^2=1-r^2$

The radius ρ of the Kaluza-Klein extra dimension has the size of $\alpha=1$. This global curvature parameter also determines the curvature of **AdS/dS**. We will later assume that α is very small (of size plank–length).

To obtain an embedding for the Schwarzschild space, set

$x_0=A\cos(\omega),\, y_0=A\sin(\omega)$ with $A^2=\rho^2+r^2-1$ and get

$$ds^2=A^2 d\omega^2+dA^2-d\rho^2-\rho^2 d\beta^2-dr^2-r^2 d\phi^2-r^2 d\Omega \tag{II.2}$$

Note: ρ, r are still normalized coordinates, scaled via $1/\alpha$ and therefore also the radius *A* of the time coordinate. In a non-normalized sphere one would have $A^2=\rho^2+r^2-\alpha^2$.

From the several possible embeddings let's choose the following

$\rho=1\Rightarrow A=r$ and $\omega=\kappa\sinh(t),\phi=\kappa\cosh(t)$ and receive

$$ds^2=(r\kappa)^2 dt^2-r^2 d\kappa^2-d\beta^2-r^2 d\Omega \tag{II.3}$$

Summarized the complete parametrization is now

$$\begin{array}{ll} x_0=r\cos(\kappa\sinh(t)), & y_0=r\sin(\kappa\sinh(t)) \\ x_i=r\cos(\kappa\cosh(t))\cdot\sigma_i, & y_i=r\sin(\kappa\cosh(t))\cdot\sigma_i\ \text{ for } i=1,2,3 \\ x_4=\cos(\beta), & y_4=\sin(\beta) \end{array} \tag{II.4}$$

where σ_i are the usual spherical unit coordinates.

Comparing the metric above with a "target" metric of the general kind

$$ds^2 = a^2(r)dt^2 - b^{-2}(r)dr^2 - r^2 d\Omega \quad , \tag{II.5}$$

leads to $\kappa = a/r$ and the equation

$$\beta'^2 = b^{-2}(1 - b^2(a' - a/r)^2) =: m^{2,} \quad (f' := df/dr) \quad . \tag{II.6}$$

(II.8)has a solution for $b^2 \geq 0$ if also $m^2 \geq 0$ or

$$1 \geq (b \cdot (a' - a/r))^{2.} \tag{II.7}$$

This embedding provides a wide class of stationary metrics, but we restrict ourselves now to Schwarzschild - De-Sitter metrics, that is to a metric with

$$\begin{aligned} &-g_{rr}^{-1} \equiv b^2 = g_{tt} \equiv a^2 = 1 + V \, , V = -r_0/r + \lambda r^2 \\ &\Rightarrow \kappa^2 = (1 - r_0/r)/r^2 + \lambda \text{ and} \\ &b \cdot (a' - a/r) = \frac{V'}{2} - \frac{(1+V)}{r} = \frac{1}{r}\left(\frac{3 r_o}{2r} - 1\right) \end{aligned} \quad .$$

Here the additional curvature constant λ appears. This constant has, as we have just seen, no influence on (II.7). But a second, arbitrary curvature constant (we have already the global α) needs to be explained again in some way later.

The relation (II.7) now simplifies to

$$r \geq \left| \left(\frac{3}{2} \cdot \frac{r_0}{r} - 1\right) \right| \quad . \tag{II.8}$$

Setting $r_0 = 0$ in (II.8) one gets the first result: $r \geq 1$. So even for the flat target space the considered embedding has a gap at the origin of the coordinate system. So the embedding implies a natural minimal structure of the space with size 1, that is the size of the global curvature α . So we assume that α is of the size of a plank length.
In addition, because of $a^2 \geq 0$, we have the constraint $r + \lambda r^3 \geq r_0$. We claim that the embedding should exist at least $\forall r \geq r_0$ and obtain from (II.8) $r_0 \geq 1/2$ as a necessary condition. Further, by simple dividing and subtracting the relation $r \geq r_0 \geq 1/2$, it is equal to

$$1/2 \geq \frac{3}{2} \cdot \frac{r_0}{r} - 1 \geq \frac{3}{4\mathrm{r}} - 1 = -r + \frac{(r - 1/2)^2 + 1/2}{r} \quad \Rightarrow r \geq \frac{3}{2} \cdot \frac{r_0}{r} - 1 \geq -r$$, which is (II.8) .

So for all $\boldsymbol{r \geq r_0 \geq 1/2}$ the embedding is possible, and any smaller value of r_0 would no more allow the radial parameter r to reach r_0 . More precisely, for $r_0 < 1/2$ and decreasing r_0 , the allowed minimal r_{min} value of r decreases first $r_{min} = \sqrt{(1/4 + 3/2\mathrm{r}_0)} - 1/2 > r_0$ until r_0 reaches the value of 1/6. Then it jumps again to ½ and increases again $r_{min} = \sqrt{(1/4 - 3/2\mathrm{r}_0)} + 1/2$ up to 1 and would increase further on, if r_0 would become negative (a repulsive Newton-Potential). As already mentioned, this inequality holds for any λ , specially also for the Schwarzschild – Anti-de-Sitter metric with $V=-r_0/r+\lambda r^2,\ \lambda>0$, which has no singularity at $r=r_0$. But even in this case, *r* could not reach any value below r_0 .

To summarize: Only for

$$\boldsymbol{r_0 \geq 1/2} \tag{II.9}$$

could the Schwarzschild space **Sch** be entirely included in this model. Again we found the value ½ as already in (I.1)**.**
For $a^2=b^2=1+V$ one has (see (II.6) ff)

$$m^2=\beta'^2=\frac{1}{(1+V)}\Big(1-\frac{1}{r^2}\Big(1+V-\frac{r}{2}\frac{dV}{dr}\Big)^2\Big)=:\frac{1}{a^2}(1-p^2) \quad , \tag{II.10}$$

and so, for the Schwarzschild – de-Sitter metrics, the phase β of Kaluza-Klein's S^1 is given by the integral

$$\beta=\pm\int\frac{dr}{r}\sqrt{\frac{r^2-(1-3\mathrm{r}_0/2\mathrm{r})^2}{1-r_0/r+\lambda r^2}} \tag{II.11}$$

As for the Kasner and Fronsdal embedding, this last primitive could not be explicitly presented, but it behaves well for all values, where the square root is defined. Much more important for interpretation is the integrand as we will see.
Extending to a complex space, β is defined everywhere. It just becomes imaginary and so the radius of the Kaluza-Klein dimension now decreases/increases (depending on the sign in the last formula).
For $r_0\geq 1/2$ the sign of $m^2:=a^{-2}(1-p^2),\ p^2\geq 0$ changes at $r=r_0$ and is positive for all $r>r_0$. Because $a^2<0$ for $0\leq r<r_0$ and $\lim_{r\to 0}p^2=\infty$ the sign of m^2 becomes again positive near the origin at $r=0$. The region of negative m^2 (or regions) form a shell (or shells) around the origin. In IV we will discuss this effect in a related context.

Now have a look at the shape of the embedding.
For constant $r\geq r_0$ and $\kappa^2=\lambda+(1-r_0/r)/r^2\neq 0$ and increasing parameter time the path on ***M*** never reaches the same point on the manifold again. The same is true for fixed time and increasing r . The space inherits this property from **dS** and the embedding is causal in this sense.

The phase κ has a maximum at $r=3/2\cdot r_0$, $\kappa=\sqrt{(\lambda+\frac{4}{27\mathrm{r}_0^2})}$. So for any

$r_a: r_0<r_a<3/2\mathrm{r}_0$ exists a $r_b>3/2\mathrm{r}_0$ such that $\kappa(r_a)=\kappa(r_b)$, and so the complex coordinates on the light cone subspace ($(x_i,y_i), i=0\ldots 3$) at (t,r_a) and (t,r_b) have the same phase and only differ in their "amplitude". This coherence may be interpreted as the coherence of two particles.
On a 2-sphere $r=\text{constant}$, with increasing parameter time, the time and space dimensions oscillate coherent, like ordinary waves. Not so the Kaluza-Klein dimensions, which only depend on r. At a singularity ($\kappa=0$), the oscillation is frozen. Going through this singularity (or even if $r_o<1/2$ for $r<r_{min}$), the oscillation starts again, but in the opposite sense. The time and radial part of the 4 -D metric changes sign and the radius of the Kaluza-Klein S^1 – sphere decreases with decreasing r . At $r=0$,

all dimensions, without the Kaluza-Klein one (the whole space-time), vanish.
The phase β in this section is selected in such a way that the initial metric becomes the desired target space. But without this there is no reason that β only depends on the radial coordinate. The question arises, what would happen, if we change β in some way. In section IV we will consider space-time under another perspective, and based on these results we will discuss metrics with a more general phase β in section V and VI.

III The complex Space.

In this section we want to consider the embedding provided above in context of Kähler geometry. The other sections are not dependent on the content of this one, so we think of this section as a proposal for people with a deeper knowledge in this subject to take a closer look at these kinds of spaces.
The 10 dimensional space carries the natural complex structure, which makes it a Kähler manifold [Hu,Mor]. The metric is the standard metric on $C^{1,4}$, a complex Lorentz (or Minkowski) metric

$$\langle u,v\rangle = u_0\cdot\bar{v}_0 - \sum_1^4 u_i\cdot\bar{v}_i,$$
$$d^2 s = \langle dz_, d\bar{z}\rangle$$

.

The manifold

$$M = \{z\in C^{1,4}: \langle z,z\rangle = -1\}.$$

is a complex de-Sitter space. As the real one it is projective. Assuming the further restriction $|z_4| = 1$, the manifold is the product of a light cone and U(1).
Usually one derives a metric on a projective space in a coordinate system, which arises from a projection onto the first coordinate z_0 (the chart $U_0 := \{(z_0 : \ldots : z_n) \mid z_0 \neq 0\}$).
For the embedding in the last section III, $|z_4|$ is always non-zero. Also, for this reason, but not only, using a projection on this coordinate, seems the better choice. This will give us the projective space $CP^{1,3}$ .The metric induced from $C^{1,4}$ via M is (see Appendix D)

$$ds^2 = \frac{1}{1-u^2}\cdot\left(\langle du,du\rangle + \frac{|\langle u,du\rangle|^2}{1-u^2}\right) \quad , \tag{III.1}$$

where $u_j = \frac{z_j}{z_4}$, for $j = 0,1,2,3$, $u^2 := \langle u,u\rangle_M$ and $\langle .,.\rangle_M$ is the complex Lorentz metric.

Metric (III.1) corresponds to the real de-Sitter metric (see [Dr]) in projective coordinates, in the same way as the Fubiny-Study metric corresponds to the real projective metric of the ball. It looks almost like the usual metric on the hyperbolic space $\mathbb{C}H^4$ [Go], but with the Minkowski scalar product <,> "inside", instead of the Euclidean[2] .
Denote the metric on the tangent bundle now as:

$$g(v,w) = \frac{1}{1-u^2}\cdot\left(\langle v,w\rangle + \frac{|\langle v,w\rangle|^2}{1-u^2}\right), \quad v,w\in T_u(M) \tag{III.2}$$

Using Einstein sum conventions, (III.1) reads as

2 Due to the definition in [Vr] it is $\mathbb{C}P^{1,3}$. . Note the opposite sign of the metrics here and in the mathematical literature.

$$d^2 s = h_{\mu\nu} du^\mu d\bar{u}^\nu, \;\; h_{\mu\nu} = g(\partial/\partial u_\mu, \partial/\partial \bar{u}_\nu) \quad ,$$

$$h_{\mu\nu} := p(\eta_{\mu\nu} + p\bar{u}_\mu u_\nu) \; , \;\; \eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1), \;\; p = (1-u^2)^{-1} \quad .$$

The metric (III.1) / (III.2) is Kähler, with Kähler potential ϕ (for definition see [Hu,Mor,Go])

$$\phi = \log(1-u^2) \; , \quad h_{\mu\nu} = \frac{d^2\phi}{\partial z_\mu \partial \bar{z}_\nu} \quad . \qquad \text{(III.3)}$$

For calculating the determinant we write the hermitian matrix h as

$$h = p\,\eta(1+p\sigma), \;\; \sigma^\mu_\nu = n^{\mu\lambda}\bar{u}_\lambda u_\nu = \bar{u}^\mu u_\nu, \;\; n^{\mu\lambda} n_{\lambda\nu} = \mathbf{1} \; (= diag(1,1,1,1)) \quad .$$

Therefore $\det h = \det(p\eta)\cdot\det(1+p\sigma) = -p^4\cdot\det(1+p\sigma)$.

The second factor is calculated using

$\det A = \exp(\mathrm{tr}\log(A))$ (tr := Trace of the matrix)

With $\sigma^2 = u^2\cdot\sigma \Rightarrow \sigma^j = u^{2(j-1)}\cdot\sigma$ we get

$$\log(1+p\sigma) = -\sum_{j=1}^{n}\frac{(-p\sigma)^j}{j} = \frac{-\sigma}{u^2}\sum_{j=1}^{n}\frac{(-pu^2)^j}{j} = \frac{\sigma}{u^2}\log(1+pu^2) \quad \text{for} \quad 0 < |u^2| < 1$$

and $\log(1+p\sigma) = p\sigma$ if $u^2 = 0$. Now $tr(\sigma) = u^2$ and so

$$\mathrm{tr}\log(1+p\sigma) = \log(1+pu^2), \;\; \text{if } |u^2| < 1 \quad .$$

Finally we get

$$\det(h) = -p^4\cdot(1+pu^2) = -(1-u^2)^{-5}$$

and so $\log(\det(h)) = \pm i\pi - 5\log(1-u^2)$.

The Ricci Tensor for Kähler manifolds [Hu,Mo] is simply given by

$$Ric_{\mu\nu} = -\frac{\partial^2\log(\det(h))}{\partial z_\mu \partial \bar{z}_\nu} \;\; \Rightarrow \;\; Ric_{\mu\nu} = 5\frac{\partial^2\log(1-u^2)}{\partial z_\mu \partial \bar{z}_\nu}$$

Comparing this with (III.3) we see, that the metric is Kähler-Einstein $Ric_{\mu\nu} = 5h_{\mu\nu}$.
The curvature is positive as it is for the classical real de-Sitter space (and so the name "complex de-Sitter space" is in all senses appropriate).
For any embedding into M endowed with the metric (III.1) and with $|z_4| := \rho = \text{constant} \Rightarrow u^2 = 1-\rho^{-2}$ (like we have done in section III with constant = 0) the metric simplifies to

$$ds^2 = \rho^2\cdot\langle du, du\rangle + |(\rho^2\cdot\langle u, du\rangle)|^2$$

and if $\rho = 1$ or after a rescaling

$$ds^2 = \langle du\,, du\rangle + \left|\langle u\,, du\rangle^2\right| \tag{III.4}$$

Because $u^2 = const \Rightarrow \langle u\,, du\rangle = -\langle du\,, u\rangle$ ($\rightarrow$ $\langle du\,, u\rangle$ is pure imaginary) one may write also

$$ds^2 = \langle du\,, du\rangle - \langle u\,, du\rangle^2 \;.$$

A consequence of this condition is that the second summand in the metric does not contributes to the Kähler form.
For writing down the Kähler form we use upper-lower indices with respect to the Lorentz metric $u^\mu = \eta^{\mu\nu} u_\nu, \ldots$ and write the metric again as

$$ds^2 = h_{\mu\bar{\nu}}\, du^\mu\, d\bar{u}^\nu\,,\;\; h_{\mu\bar{\nu}} = (\eta_{\mu\nu} + u_\mu \bar{u}_\nu) \;\;.$$

The Kähler form is now (up to some factor)

$$\omega = h_{\mu\nu}\, du^\mu \wedge du^\nu = du_\mu \wedge d\,\bar{u}^\mu + u_\nu \bar{u}_\mu\, du^\mu \wedge d\,\bar{u}^\nu = du_\mu \wedge d\,\bar{u}^\mu - (u_\mu d\,\bar{u}^\mu) \wedge (u_\nu d\,\bar{u}^\nu) \;\;.$$

Obviously, the second summand vanishes. So the Kähler form is just the same, as for the flat metric (the complex Lorentz metric).

$$\omega = du_\mu \wedge d\,\bar{u}^\mu = dv_\mu \wedge dw^\mu\,,\quad u = v + iw\,,\; v\,, w \in \mathbb{R}^{1,3}$$

An embedding is called totally real (respectively Lagrangian in symplectic geometry), if the induced Kähler form vanishes everywhere [3] (Go, Oh). So any totally real embedding into the complex Minkowski space $C^{1,3}$, for which u^2 is constant, is also totally real as embedding into **M** equipped with the metric (III.1).
The embedding provided in the previous section is not Lagrangian. Its induced Kähler form doesn't vanish. For constructing an Lagrangian embedding set

$$\begin{aligned}
&u_0 = a\,e^{i\omega} \Rightarrow v_0 = a\cos(\omega), w_0 = a\sin(\omega),\quad a\,, \omega \in \mathbb{R}\\
&u_j = b_j e^{i\eta} \Rightarrow v_j = b_j\cos(\eta), w_j = b_j\sin(\eta),\;\; b_j\,, \eta \in \mathbb{R}\,, j = 1{,}2{,}3\\
&b^2 := \sum b_j b_j\,,\;\; a^2 = b^2 + k^2\,,\;\; k^2 := u^2 = constant\,,\;\; b := \sqrt{(b^2)}
\end{aligned} \tag{III.5}$$

and therefore

$$dv_0 \wedge dw_0 = a\,da \wedge d\,\omega\,,\;\; \sum dv_j \wedge dw_j = \sum b_j\, db_j \wedge d\,\eta = 1/2 \cdot \sum db_j^2 \wedge d\,\eta = b\,db \wedge d\,\eta \;\;.$$

Because $b\,db = 1/2\, d\,(b^2) = 1/2\, d\,(a^2) = a\,da$ the Kähler form is $\omega = a\,da \wedge d\,(\omega - \eta)$ and so the Kähler form vanishes, if $\omega - \eta = f\,(a)$, which is what we demand in the following. For calculating the metric (III.4) under this restriction, rename $b = r$ (for getting the "familiar" 2-D surface term $r^2 d\,\Omega$ in the metric) and receive

3 For a totally real embedding the induced metric has no imaginary part. On the other hand there are complex submanifolds (closed complex subspaces). If, for an embedding, the tangential space at a point is mapped into a complex subspace , this point is called a complex point. The Kähler angle (or angle of holomorphy, [Go, Sc]) between two vectors measures this difference. It is always $\pi/2$ for totally real embeddings and zero at complex points.

$$du_\mu d\bar{u}^\mu = dv_\mu dv^\mu + dw_\mu dw^\mu =$$
$$= a^2 d\omega^2 - r^2 d\eta^2 - r^2 d\Omega = k^2 d\omega^2 + r^2(d\omega^2 - d\eta^2) - r^2 d\Omega \quad .$$

With $u_0 d\bar{u}_0 = a(da + ia\, d\omega)$, $\sum_{j=1}^{3} u_j d\bar{u}_j = \sum_{j=1}^{3} r_j (dr_j + ir_j d\eta) = r(dr + ird\eta)$, equation (III.4) finally becomes (use $ada = rdr$ for constant u^2)

$$ds^2 = a^2 d\omega^2 - r^2 d\eta^2 - r^2 d\Omega + (a^2 d\omega - r^2 d\eta)^2 \quad . \tag{III.6}$$

The metric now diagonalizes with the substitution

$$d\psi = \frac{a^2+1}{a^2+1-r^2}(d\omega - d\eta) + d\eta$$

This DGL could always be integrated, due to $\omega - \eta = f(r)$. From the substitution one also immediately sees that $\psi - \eta$ is also just a function of ***r,*** and vice versa, r is just a function of $\psi - \eta$. Summarized, diagonalizing (III.6) yields

$$ds^2 = a^2 \frac{1+a^2-r^2}{1+a^2}((1+a^2-r^2) d\psi^2 - (r/a)^2 d\eta^2) - r^2 d\Omega \tag{III.7}$$

Now if $a = r \Leftrightarrow k = 0$, (as in section III; the condition $k^2 := u^2 = 0$ means, that we consider the complex light cone) equation (III.7) reduces to

$$ds^2 = \frac{r^2}{(1+r^2)}(d\psi^2 - d\eta^2) - r^2 d\Omega \quad . \tag{III.8}$$

To calculate the geodesics of this metric, we use the Euler-Lagrange formalism with $\dot{x} := dx/dt$, and for an affine parameter t proportional to the arc length

$$S[x, \dot{x}] = \int \sqrt{(L(x, \dot{x}))}\, dt = min \Leftrightarrow S[x, \dot{x}] = \int L(x, \dot{x})\, dt = min$$ holds [ST].

So the Lagrange function corresponding to (III.8) is

$$L = \frac{r^2}{(1+r^2)}(\dot{\psi}^2 - \dot{\eta}^2) - r^2 \cdot S, \ \text{ with the usual surface term } S = \dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2 \quad .$$

Writing $r = \sinh(\varsigma)$, where ς is some function of $\psi - \eta$, results in $r' := \frac{dr}{d\psi} = -\frac{dr}{d\eta}$, and the Euler-Lagrange equations for the variables ψ and η are

$$\frac{d}{dt}(\cosh^2(\varsigma)\cdot\dot{\psi}) = \cosh(\varsigma)\sinh(\varsigma)\cdot\varsigma'(\dot{\psi}^2 - \dot{\eta}^2 - S)$$
$$\frac{d}{dt}(\cosh^2(\varsigma)\cdot\dot{\eta}) = \cosh(\varsigma)\sinh(\varsigma)\cdot\varsigma'(\dot{\psi}^2 - \dot{\eta}^2 - S)$$

and so it follows

$$\cosh^2(\varsigma)\cdot(\dot{\psi} - \dot{\eta}) = const \quad .$$

Because ς is only a function of $\psi-\eta$, the last equation is solvable iff $\psi-\eta$ is constant. So on geodesics we have $r=constant$. Now putting this back into the metric (III.8) only the surface term remains. The corresponding geodesics are the great circles on the ball of radius r. Because $\psi-\eta$ is constant also $\omega-\eta$ is constant and this geodesics may be written as (using the equator as a great circle)

$$u_0=r\,\mathrm{e}^{i(\omega(t)-\omega_0)},\ u_1=r\cos(\varphi(t))\mathrm{e}^{i\omega(t)},\ u_2=r\sin(\varphi(t))\mathrm{e}^{i\omega(t)},\ u_3=0$$

or $$u_0=r\,\mathrm{e}^{i(\omega(t)-\omega_0)},\ u_1=u_0\cos(\varphi(t)),\ u_2=u_0\sin(\varphi(t)),\ u_3=0\ .$$

The phase difference ω_0 is the value of the function $f(r)=\omega-\eta$ at constant r.
On the geodesic great circles any point is oscillating "wave-like" in higher dimensions. The curves are space-like and nothing like an "eigentime interval" or a "proper length" exists. The metric degenerates to the one of a 2-D surface, as it is for the light cone in a flat Minkowski space ($t^2=r^2 \Rightarrow ds^2=-r^2 d\Omega$). There is no dependency on the geodesics between the point on the surface (the angle φ on the great circle) and the frequency ω. So for example one may set $\varphi=constant$ to get "oscillating" points or $\phi=\omega$ for stationary waves.
Summarized we have, that the geodesics of an Lagrangian embedding into the complex light cone ($u^2=0$) with coherent phases of the space-like dimensions, are arbitrary "waves" on the Minkowski light cone.
Now we calculate the volume form of the Lagrangian subspace (again for any *k*) . We have

$$u_j=x_j e^{i\eta} \Rightarrow e^{-2i\eta}du_j\wedge du_k = dx_j\wedge dx_k+i\left(x_j dx_i+x_i dx_j\right)$$ and therefore
$$e^{-3i\eta}du_1\wedge du_2\wedge du_3 = dV+idS\wedge\eta$$

where dV is the 3D volume form and dS the surface form,

$$dV := dx_1\wedge dx_2\wedge dx_3,\quad dS=x_3 dx_1\wedge dx_2+x_2 dx_3\wedge dx_1+x_1 dx_2\wedge dx_3 \Rightarrow dr\wedge dS=rdV\ .$$

So, with $du_0=e^{i\omega}(da+i\,a\,d\omega),\ a^2=r^2+k^2,\ \omega-\eta=f(r)$ the volume form is

$$\begin{aligned}
dVol &:= du_0\wedge du_1\wedge du_2\wedge du_3\\
&= e^{i(\omega+3\eta)}\left(ia\cdot d\omega\wedge dV+i\,da\wedge dS\wedge d\eta-a\cdot d\omega\wedge dS\wedge d\eta\right)\\
&= e^{i(\omega+3\eta)}\left(i\left(a\,d\omega-\frac{r^2}{a}d\eta\right)\wedge dV+a\,d\eta\wedge d\omega\wedge dS\right)\\
&= a\,e^{i(\omega+3\eta)}\left(rf'+i\left(\frac{k}{a}\right)^2\right)d\eta\wedge dV.
\end{aligned}$$

We see that the volume form is identical zero if $\omega-\eta=f(r)$ = constant and k = 0. In this case the metric (III.7) degenerates again just to a 2-D surface, the space is a light cone. From the volume form we read of the Lagrange angle (the phase of the form) [Vr,An] as $\omega+3\eta+\mathrm{atan}\left(k^2/(rf'a)\right)$. A Lagrangian submanifold is minimal, iff this angle is constant [An]. But then from $\omega-\eta=f(r)$ it immediately follows, that ω and η could only depend on ***r*** and so the volume form vanishes also. In this case, an appropriate choice of the arbitrary function $f(r)$ would transform (III.7) to a 3D flat space metric. So the minimal Lagrangian submanifolds with $u^2=const$, are the light cone (with all metric structure in 2-D spheres) or the pure 3-D spatial space (the absence of "time-intervals" could be interpreted as anything "travels with infinite speed").

In complex coordinates, the parametrization (II.4) looks like

$$\begin{aligned}&u_0=r\,e^{i\,\omega},\;\; u_j=x_j\,e^{i\eta},\, j=1,2,3,\;\; u_4=e^{i\beta(r)}\\&\omega=\kappa(r)\sinh(t),\;\eta=\kappa(r)\cosh(t)\end{aligned}\quad, \tag{III.9}$$

which corresponds in the projective space (omitting the 2-D spherical part) to

$$u_0=r\,e^{i\,\omega},\;\; \omega=\kappa(r)\sinh(t)-\beta(r) \qquad u_r=r\,e^{i\eta},\;\; \eta=\kappa(r)\cosh(t)-\beta(r) \quad .$$

Here the difference of the angles is $\omega-\eta=-\kappa\,e^{-t}$, which vanishes for $t\to\infty$. So in this limit, the embedding becomes a minimal Lagrangian embedding, the light cone. Anything is concentrated on 2-D spheres, the boundary of the space. Remember, this result holds true for any stationary, isotropic metric, for which (II.7) is true. That "anything is concentrated" on a 2-D surface reminds us that the entropy of a black hole depends only on the surface area.

IV Geodesics from Classical Kaluza-Klein Theory

The classical K-K theory

Up to now the Kaluza-Klein dimensions have been only used to result in the correct metrics, but not in their original sense to combine gravitation theory with electro-magnetism. First have a look at the standard interpretation. For this we rename the five coordinates as usual to $x^\mu,\ \mu=0\ldots3,\ y:=x^4$ (sometimes the index 5 is used for the extra dimension) and use again the standard notation, i.e summation over repeated upper, lower indices from 0 to 3 and $a_\mu:=g_{\mu\nu}a^\nu$, where $g_{\mu\nu}$ are the components of a 4-D metric tensor.
The Kaluza-Klein line element is [WL, Le, Str, Bl, Du]

$$ds^2 = g_{\mu\nu}\,dx^\mu\,dx^\nu-\Phi^2\cdot(dy+A_\mu\,dx^\mu)^2 \tag{IV.1}$$

and the corresponding Lagrange function

$$L = g_{\mu\nu}\,\dot{x}^\mu\,\dot{x}^\nu-\Phi^2(\,\dot{y}+A_\mu\,\dot{x}^\mu)^{2,}\ \text{where}\ \dot{x}:=dx/ds \tag{IV.2}$$

Now make the essential assumption of the theory, that the fields (A_μ and Φ) do not depend on y (called the "cyclic" assumption). So the fifth coordinate y is cyclic and due to Noether's first theorem, there is a conservation law. The geodesic equation for this coordinate just becomes

$$\Phi(\,\dot{y}\,+\,A_\mu\,\dot{x}^\mu) = q \;\;\Rightarrow\Phi(\,dy\,+\,A_\mu\,dx^\mu)=q\,ds \tag{IV.3}$$

where *q* is an integration constant (identified later as the ratio unit-charge per mass). Setting $\Phi=1$, calculating the other geodesics, and using (II.6) leads to the familiar equation of a particle in an electromagnetic field [Bl,St].

$$\begin{aligned}&\ddot{x}^\sigma+\Gamma^\sigma_{\mu\nu}\dot{x}^\mu\,\dot{x}^\nu=q\cdot F^\sigma_\mu\,\dot{x}^\mu\quad\text{or with}\,u^\mu:=\dot{x}^\mu\\&Du^\sigma:=\frac{du^\sigma}{ds}+\Gamma^\sigma_{\mu\nu}u^\mu\,u^\nu=q\cdot F^\sigma_\mu\,u^\mu\end{aligned} \tag{IV.4}$$

A main problem of the Kaluza-Klein (or KK) theory is that the metric is in fact not really a metric, as the quantities A_μ are not coordinates of a four-vector. As the components $A_\mu=g_{4\mu}/g_{44}$ are functions of the metric components, their behavior under coordinate transformation differs essentially from that of coordinates of a four-vector. If the A_μ

are defined to be vector-components, the metric "is not invariant under arbitrary 5-D coordinate transformations" [Bl], which means, it is not a metric! To bring this difference in line, one limits the allowed coordinate transformations. One demands that they must have the specific form $x^\mu \to \tilde{x}^\mu(x^\mu),\ \ y \to \tilde{y} = y + h(x^\mu)$, for which g_{44} remains unchanged and so

$$A_\mu dx^\mu = g_{4\mu} dx^\mu / g_{44} \to \tilde{g}_{4\mu} d\tilde{x}^\mu / g_{44} + (\partial h / \partial x^\mu) d\tilde{x}^\mu = \tilde{A}_\mu d\tilde{x}^\mu + d\tilde{y} - dy \quad .$$

So under this kind of transformation, the "vector" $A_\mu dx^\mu$ "remains the same" in different coordinate systems modulo some gauge transformation, which does not affect electromagnetic theory. But, applying mathematical theory and methods on not well-defined "hybrids" like this one, is in some way suspect and must be done very carefully.

Equation (II.6) is usually interpreted in correspondence to gauge invariance. If applying a gauge transformation, the trajectory through the 5'th dimension has to be changed to keep the expression constant. The overall change may be interpreted as a coordinate transformation of the allowed type.
In standard reading of the KK theory, the constant q in (IV.3) is interpreted as the ratio e-charge/mass. But in the geodesic equations (IV.4) the derivatives, denoted by the dot, are with respect to the total five dimensional arc length and this is, due to (IV.1), different from the 4-dimensional arc length $ds_4 = d\tau$ (the eigentime). Under the cyclic assumption with (IV.3) we have $ds^2 = d\tau^2 - q^2 ds^2 \ \Rightarrow ds^2 \cdot (1+q^2) = d\tau^2$ (see also [WL]).
To compare (IV.4) with standard electromagnetic theory, we have to replace in (IV.4) the derivative with respect to the arc-length s , with the derivative with respect to the 4-D arc length τ ($ds \to d\tau$). Doing this, the structure of equation (IV.4) keeps the same, if we define the dot as the derivative with respect to τ and if we replace $q \to q/\sqrt{(1+q^2)}$, which now has to be interpreted as the ratio e-charge/mass.
The "covariance"-problem of the theory arises from the 5'th dimension and the interpretation of A_μ as components of a four vector. So one should try to find an embedding of the 4-D space into the 5-D Kaluza-Klein space, with the desired properties, that is, in the optimal case, with the same geodesic equations. The development and analysis of a satisfying, "covariant" Kaluza-Klein Theory, which considers a "full dependency" on all dimensional coordinates, is a special field of research (see e.g [WL, Le]) . Such an ansatz results in generally "orthogonal forces" (orthogonal to the 4-D hyperplane), looses gauge invariance, and results also in some other interesting phenomena (see [Le] and literature cited there). [Le] constructs a theory in which the forces are again gauge invariant, but it would not be astonishing (for me), if in a "final" theory, gauge invariance must be dropped. As seen, "using some gauge" has in this theory something to do with choosing a coordinate system, but in general relativity a lot of other, "classically conserved" quantities depend on the coordinate system. Noether's second theorem does not allow conserved tensor components in general relativity [BB,FFM,No]. I not want to discuss and go deeper inside the research about KK-Theory. A common effect of all these approaches is also that electromagnetic and gravitation forces are not independent, even though the authors do not mention this explicitly.

Geodesics from a KK theory– like ansatz

The model discussed in this article seems to not fit together with the standard KK-Theory. but nevertheless it may be of interest to see how it looks under a similar point of view.
For this let's come back to the metric (II.3) with coordinates (II.4). With the notation of equation (II.6) but now first keeping β as independent coordinate. So the metric is

$$ds^2 = a^2 dt^2 - b^{-2} dr^2 - r^2 d\,\Omega + m^2 dr^2 - d\,\beta^2 \qquad \text{(IV.5)}$$

and the Lagrange function, written in the usual way

$$L = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + (m\dot{r})^2 - \dot{\beta}^2 \quad .$$

The metric g is here the target metric (II.5) in section II after applying (II.11)
Let now $\dot{\beta} = B_\mu \dot{x}^\mu + B_4 \dot{y}$. In the following we use Latin indexes to run from 0 to 4 and with the additional definitions

$$B_j := \beta_{,j} \; (\; j=0\ldots4), B := B_j \dot{x}^j \equiv \dot{\beta}$$
$$P_j := \delta_{rj} m, \; P := P_j \dot{x}^j = m\dot{r}$$

(an index "r", e.g. B_r , denotes the index corresponding to a radial quantity and so on), the Lagrangian simplifies to

$$L = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - B^2 + P^2 \qquad \text{(IV.6)}$$

For an expression $X = X_j \dot{x}^j$ we have $\partial X / \partial \dot{x}^j = X_j$ and $\partial X / \partial x^j = X_{,j}$, so

$$\frac{d}{ds} \frac{\partial X}{\partial \dot{x}^j} - \frac{\partial X}{\partial x^j} = \dot{X}_j - X_{,j}$$

and the Euler-Lagrange equations are

$$\gamma_j = B(\dot{B}_j - B_{,j}) + \dot{B} B_j + PP_{,j} - \frac{d}{ds}(PP_j), \;\; \gamma_j := Du_j \;\text{ for }\; j<4, \gamma_4 = 0$$

(with Du_j as defined in (IV.4)) . Because of $B_{j,k} = B_{k,j} \Rightarrow \dot{B}_j = B_{,j}$, in the geodesic equations above no electromagnetic field will appear. The field is, as also seen obviously in the definition, a pure gauge field. With the standard KK-Ansatz $B_\mu = \Phi A_\mu, \; B_4 = \Phi$, this last symmetry reads as $\Phi (A_{j,k} - A_{k,j}) + (\Phi_{,k} A_j - \Phi_{,j} A_k) = 0$, and this may be interpreted that the field tensor ($F_{jk} = A_{j,k} - A_{k,j}$) annihilates with the second summand. The geodesic equations now reduce to

$$\gamma_j = \dot{B} B_j + PP_{,j} - \frac{d}{ds}(PP_j) \quad .$$

For the derivatives of P one calculates

$$PP_{,j} = \frac{(w^2)_j}{2} = \frac{(\dot{r})^2}{2}(m^2)_{,j} = \dot{r}^2 m m' \delta_{rj}, \;\; \frac{d}{ds}(PP_j) = \frac{d}{ds} m^2 \dot{r} \cdot \delta_{rj} .$$

and so for the Euler-Lagrange equations

$$\gamma_j = B_j \dot{B} + (\dot{r}^2 m m' - \frac{d}{ds} m^2 \dot{r}) \delta_{rj} = -m(\frac{d}{ds} m \dot{r}) \delta_{rj}$$

or

$$\ddot{x}^\sigma + \Gamma^\sigma_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = B^\sigma \dot{B} - m(\frac{d}{ds} m \dot{r}) g^{r\sigma}$$
$$\text{and} \qquad 0 = B_4 \dot{B} \tag{IV.7}$$

The last equation has the solutions $B_4=0$ and $\dot{B}=0$ (or both vanishing). The solution with $B_4=0, \dot{B}\neq 0$ splits itself into a set of different types of geodesics, and specifically contains , with $B_j=0$ for $j\neq r$ and $B_r=m.$, the geodesics of the target metric. In this last case, β may be interpreted as a gauge field, which forces trajectories to stay in the submanifold, defined by $d\beta = m\,dr$ or, e.g. if the target metric is the flat one, the field β "keeps the space flat". We will discuss other possible geodesics of type $B_4=0$ below. While for $B_4=0$ the path in the y-dimension is arbitrary, that means this dimension is artificial and superfluous, this is not the case for the solutions $B = \dot{\beta} = const.$ and. Submanifolds, created by sets of geodesics are by construction geodesic submanifolds. In the next section we will discuss the metrics of some of them.

We end this section with one final remark.
Where in the approach above, the electromagnetic field tensor always vanishes (pure gauge fields), the complex metric (III.4) offers the possibility to define a non-vanishing one in the metric via

$$\langle A, dx\rangle = i\langle u, du\rangle \Rightarrow A_\mu = i\cdot u_\nu \frac{\partial \bar{u}^\nu}{\partial x^\mu}$$
$$ds^2 = \langle u, u\rangle + \langle A, dx\rangle^2 .$$

With the parametrization of (III.5) one gets for the field

$$u_0 \frac{\partial \bar{u}^0}{\partial x^\mu} = x_\mu - \delta^0_\mu x_0 - ir\,\omega_{,\mu}, \quad \sum_{j=1}^{3} u_j \frac{\partial \bar{u}^j}{\partial x^\mu} = x_\mu - \delta^0_\mu x_0 - ir\,\eta_{,\mu}$$
$$\Rightarrow A_\mu = r\cdot(\omega-\eta)_{,\mu} \Rightarrow \langle A, dx\rangle = r\cdot(\omega-\eta)_{,\mu} dx^\mu = d(r\cdot(\omega-\eta)) - (\omega-\eta)\,dr ,$$

so this is a radial field, modulo a gauge field.

V Metrics of Submanifolds

V.1 "Almost Flat" and "Almost Schwarzschild" metrics

Before considering special submanifolds, which arise from the geodesics in the last section, we point out some general facts. For this we first review the concept of the "proper length" or "proper distance". In special relativity the "proper length" is the length with respect to the rest frame of the object, which is measured. Consequently the proper distance between two objects has to be measured in a frame, where both objects are at rest, regardless if they have a relative movement to each other or not. This is the frame of a static or stationary observer. So for example, the proper distance sun-earth is one astronomical unit, even if a satellite traveling from earth to the sun measures a different distance. For a general static (radially symmetric) metric

$$ds^2 = g_{tt} dt^2 - g_{rr} dr^2 - r^2 d\Omega \tag{V.1}$$

the proper distance l is given as $l: dl^2 = -g_{rr} dr^2$ [MT,G] and is the negative arc length $-ds^2$ of the space-like distance at synchronous time ($dt/ds=0$). Besides the proper radial distance we have a proper radial velocity v , which is the radial velocity as a static observer measures, $v^2 = g^{tt}\left(\frac{dl}{dt}\right)^2$, $g^{tt} := g_{tt}^{-1}$ [MT,Oi]. Using these relations, a straight line element($d\,\Omega = 0$) is just

$$ds^2 = g_{tt}(1-v^2)dt^2 \quad . \tag{V.2}$$

If g_{tt} is constant (let use 1 without loss of generality) , this is just the line element of special relativity. Moreover for light-like geodesics we get $dt^2 = -g_{rr}dr^2 = dl^2$ and so $v = dl/dt = 1$. So for the static observer, who measures time and length, anything looks just like special relativity, despite that the space is not flat. We will call this type of metrics and spaces **"almost flat" (AF)**, or **"special relativity – like" (SR-like)**. But of course there are differences to a flat space; for example measuring the circumference of a circle centered at $r = l = 0$, which is $2r\pi$ and not $2l(r)\pi$. But even this basic feature must be and will be discussed later. Another difference is that the Einstein tensor, and so via Einstein equation the energy – momentum tensor, generally does not vanish. With the results in [Dr 8.31 ff] (and $\exp(\mu) = -g_{rr}$, $\exp(\nu) = g_{tt}$) one arrives at

$$G_0^0 = \frac{1}{r^2}\frac{d}{dr}\left(r\left(Q^{-2}-1\right)\right),\; G_r^r = \frac{\left(Q^{-2}-1\right)}{r^2},\; G_2^2 = G_3^3 = \frac{1}{2r}\frac{d}{dr}\left(Q^{-2}\right) \quad . \tag{V.3}$$

From this immediately follows $G_0^0 = \sum_{j=1}^{3} G_j^j$, which is, if all components are positive or zero the strong energy condition [G][4]

$$G_0^0 \geq 0 \;\wedge\; G_0^0 \geq G_j^j \;\; \forall\, j = 1,2,3 \;\wedge\; G_0^0 \geq \sum_{j=1}^{3} G_j^j \quad ,$$

which is interpreted as an indication for "ordinary matter" [G,MT] . Conversely, if all components are zero or negative, we get $G_0^0 \leq 0,\; G_0^0 \leq G_j^j \;\; \forall\, j = 1,2,3$ and so G lacks any energy-condition. A metric, which looks like a **SR**-metric, usually is a vacuum, so it should not look like a metric with matter. In a vacuum there is nothing we can identify with positive energy. So we only accept **SR-like** metrics, if the Einstein tensor does <u>NOT</u> fulfills any energy-condition. We call these last inequalities

$$G_0^0 \leq 0 \;\wedge\; G_0^0 \leq G_j^j \;\; \forall\, j = 1,2,3 \;\wedge\; G_0^0 \leq \sum_{j=1}^{3} G_j^j$$

the "**vacuum energy condition**". Further we expect that asymptotically the vacuum should become a vacuum in the normal sense, that is G vanishes at infinity, which also implies that the space is asymptotically Ricci-flat ($G = 0 \Rightarrow tr(G) = 0 \Rightarrow tr(Ric) = 0$). Let's make the following definition:

Definition:
A **vacuum** is a domain in space-time, where the metric is **almost flat** (or **SR-like**) and the **vacuum energy condition** holds with $\lim_{r\to\infty} G(r) = 0$. In a stronger form of the definition, we demand additionally $G \leq 0$, i.e. $G_j^j \leq 0 \;\; \forall\, j$.

If $\lim_{r\to\infty} Q(r)$ is constant, $\lim_{r\to\infty} G(r) = 0$ holds and asymptotically for large 3-D volumes

4) Note [G] uses upper lower indices with respect to the Minkoswki metric η_{ij} and so his energy-momentum tensor is $T_{ij} = \eta_{il} G_j^l$!

$$E=\int Tr(G)\sqrt{(-det(g))}\,dx^3=8\pi\int_0^R\left(\frac{d}{dr}(r(Q^{-2}-1))\right)Q\,dr \approx -C_0-C_1\cdot R \quad .$$

The divergence is just linear, whereas for a constant Einstein tensor (as from a cosmological constant) one gets a cubic divergence.
Sometimes space-time domains, which fulfill no energy condition, are called "exotic matter". We do not use this terminology and/or interpretation in the following until we come back to it in a quite different way in section VI .

We now return to the geodesic equations. With $g_{tt}=b^2(r),\ -g_{rr}=Q^2(r)/b^2(r)$, the Lagrange function associated with the metric (V.1) is

$$L = b^2\dot{t}^2-(Q/b)^2\dot{r}^2-r^2(\dot{\theta}^2+\sin^2(\theta)\dot{\phi}^2) \quad . \tag{V.4}$$

The dot (e.g. in $\dot{r}$) denotes the derivative with respect to an affine parameter σ along the geodesic, where for time-like geodesics σ could be identified with the arc length (or the eigentime). So from $ds^2=(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)d\sigma^2$, $\mathcal{L}=g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ one gets for time-like geodesics, using $\sigma=s$, $\mathcal{L}=const.=1$ and with $ds^2=0$ for light-like (null-geodesics) $\mathcal{L}=0$ [St]. The geodesic equations for the angular and the time parameter have the known standard form and solutions $\theta=\pi/2,\ r^2\dot{\phi}=M=const.$ and $b^2\dot{t}=A=const$ and one gets the equation

$$\frac{A^2}{b^2}-Q^2\frac{\dot{r}^2}{b^2}-\frac{M^2}{r^2}=\mathcal{L}\,,\ \ \mathcal{L}=0{,}1.\ \ A^2\geq 1 \quad . \tag{V.5}$$

This equation is a necessary condition for the radial Euler-Lagrange equation to hold and for $\dot{r}\neq 0$ it is also sufficient (proven by differentiating with respect to the parameter along the curve). So for $\dot{r}\neq 0$ it can be used instead of the radial Euler-Lagrange equation. For radial geodesics through the origin ($M=0$) one gets $Q^2\dot{r}^2=A^2-\mathcal{L}b^2$.
Inserting this equation into the definition for the velocity v one obtains

$$v^2=b^{-2}\left(\frac{dl}{dt}\right)^2=\frac{1}{b^2}\left(\frac{Q}{b}\frac{dr}{d\sigma}\frac{d\sigma}{dt}\right)^2=Q^2\frac{\dot{r}^2}{A^2} = 1-\mathcal{L}\frac{b^2}{A^2} \quad . \tag{V.6}$$

For light-like geodesics ($\mathcal{L}=0$) one uses $\sigma=t$ and so has $A=1$ and again arrives at $v^2=1$, but now for any function $b^2(r)$, that is for any metric of type (V.1) . For $b^2=1$ and time-like ($\mathcal{L}=1$) geodesics, the last equation defines A via v as $A^2=(1-v^2)^{-1}$. Inserting this expression again in (V.5 with $M=0$ leads first to $Q^2\dot{r}^2=(1-v^2)^{-1}v^2$ and with $d\sigma=ds=\sqrt{(1-v^2)}dt$ and the proper length l finally to

$$\left(\frac{dl}{dt}\right)^2=\left(Q\frac{dr}{dt}\right)^2=v^2 \tag{V.7}$$

For light-like geodesics we receive exactly the same equation, but of course with $v=1$, and so (V.7) holds for all **AF** or **SR-like** spaces.
If $\lim_{r\to\infty}Q(r)=\text{constant}$ and $\lim_{r\to\infty}b(r)=\text{constant}$ we determine A^2 in (V.6), by defining initial conditions $\dot{r}=0,\text{for } r\to\infty$, and have for time-like geodesics

$$A^2 = \lim_{r\to\infty} b^2(r) \qquad \text{(V.8)}$$

(for light-like geodesics this initial condition makes no sense, we have always $A^2=1$). Next we look at circular geodesics, that is at geodesics with $\dot{r}=0$. From (V.4) one gets for the resulting radial Euler-Lagrange $\partial L/\partial dr=0$ equation $\frac{A^2}{b^4}\frac{db^2}{dr}=2\frac{M^2}{r^3}$ and after applying (V.5

$$(\mathcal{L}+\frac{M^2}{r^2})\frac{db^2}{dr}=2b^2\frac{M^2}{r^3} \quad . \qquad \text{(V.9)}$$

The equation immediately shows that for **AF** spaces ($b^2=1$) no circular geodesics exists (M has to vanish), as they don't exist for usual flat spaces . We now define an **almost Schwarzschild** (**AS)** metric as a metric with $b^2=1-r_h/r\,,\ r_h\geq 0=\text{constant}$, which for $r_h=0$ is a **vacuum** as defined above. Any radially symmetric, stationary metric with $g_{tt}=b^2$ of the desired form is for $r_h=0$ an **AF** metric, but not necessarily a **vacuum**. The parameter r_h is the event horizon of the metric in the same meaning as for standard Schwarzschild spaces. For $r\rightarrow r_h$ time dilation dt/ds becomes infinite and behind it one has $\dot{t}<0$ ("time moves backward") and the metric changes its signature. For **AS** metrics (V.9) becomes

$$(\mathcal{L}\frac{r^2}{M^2}+1)\frac{r_h}{r}=2(1-\frac{r_h}{r}) \overset{\mathcal{L}=0}{\Rightarrow} \frac{r_h}{r}=\frac{2}{3} \Rightarrow b^2=\frac{1}{3} \quad , \qquad \text{(V.10)}$$

and inserting this in (V.5 we get for circular null - geodesics ($\mathcal{L}=0 \rightarrow A=1$)

$$M=\frac{r}{b}=\frac{3}{2}\sqrt{3}\,r_h \ ,\frac{d\phi}{dt}=\frac{M}{r^2}=\frac{2}{\sqrt{3}\,r_h}, E_M=\frac{M^2}{r^2}=3 \quad .$$

For **AS** spaces the limit value in (V.8) is one, so we have with (V.6) for straight timelike geodesics to the origin ($M=0$) with initial conditions $\dot{r}=0, \text{for } r\rightarrow\infty$

$$v^2=b^{-2}(\frac{dl}{dt})^2=(Q\dot{r})^2=r_h/r \quad (\ v^2=(Q\dot{r})^2=1 \ \text{ for light-like}). \qquad \text{(V.11)}$$

The right-hand side r_h/r of the equation is defined even for $r<r_h$ and hence should be any expression at the left side. But b^2 becomes negative and so dl/dt has to be imaginary. At $r=r_h$, dl/dt has to be zero, in order that the expression keeps finite, whereas $dl/dr=Q^2/b^2$ goes to infinity, if Q^2 has no zero there. The "concept" of proper length lacks inside the event horizon. For this, the area $r<r_h$ is a gap or a single point. This is nothing special in this more general metric, this is the same in the usual theory.

We now consider the Newton limit of our equations for time-like curves. For (V.10) we receive the familiar standard result $r_h/2=M^2/r$ for an orbit around a mass of $r_h/2$, if we use a gravitation constant = 1. But equation V.11, the straight geodesics through the origin, becomes in this limit $Q_\infty^2 \cdot (dr/dt)^2 = r_h/r$, $Q_\infty^2 := \lim_{r\to\infty} Q(r)$. After differentiating Newtons law we get $\ddot{r} = -Q^{-2} r_h/2 \cdot r^{-2}$, the same result as for a mass of $Q_\infty^{-2} r_h/2$. So, if $Q_\infty \neq 1$, orbital and straight-directed forces look different, in the same way as their sources would have different masses, or more familiar, the gravitation constants in both equations are different. This is a consequence of the nature of g_{rr} and hence of the **vacuum.** $-g_{rr}$ does not become one in the Newton limit, and consequently the proper length $dl = Q_\infty dr.$ will not match with the parameter length. So even for the velocities in the vacuum we get in the newton limit $v^2 = dl/dt \neq dr/dt$. How this could be interpreted, we will discuss later. We will call the gravitation constant, coming from the straight forward movement, the **Newton gravitation constan**t γ_N , because it seems closer to Newtons "falling apple" example, and the one in the orbital forces the **Kepler gravitation constant** γ_K , where $\gamma_K/\gamma_N = Q^2$. In normalized units we still use $\gamma_N = 1$ and so we have $\gamma_K = Q^2$.

As for the usual Schwarzschild metric, we can define a coordinate transformation into the system of a geodesic observer [St] and receive an associated LeMaitre-metric

$$\begin{aligned} dT &= dt + \sqrt{\frac{r_h}{r}} \frac{Q^2}{b^2} dr, \quad dR = dT + \sqrt{\frac{r}{r_h}} dr = dt + \sqrt{\frac{r}{r_h}} \frac{Q^2}{b^2} dr \\ ds^2 &= dT^2 - \frac{r_h}{r} dR^2 - r^2 d\Omega \end{aligned} \tag{V.12}$$

The DGL's have the solutions

$$T = t + \int \sqrt{\frac{r_h}{r}} \frac{Q^2}{b^2} dr, \quad r = \left((R-T)\frac{3}{2}\sqrt{r_h}\right)^{2/3} \quad .$$

In this coordinate system, there is no difference to the usual theory with an event horizon of r_h .

A special "almost flat" metric.

The first type of submanifolds which arise from the geodesics in the last section of chapter IV and which we will consider now, occur from the solution $B = const.$ of (IV.7). This is the usual solution, received in standard KK theory and interpreted there as electric charge conservation [Du,Str,Bl]. The remaining other (4-D) geodesic equations of (IV.7) are then just the geodesics of the metric (IV.5) for β is constant. If the target metric is the flat metric, using (II.6) or (II.10) with $a = b = 1 \Rightarrow m^2 = 1 - 1/r^2$, one gets from (IV.5)

$$ds^2 = dt^2 - r^{-2} dr^2 - r^2 d\Omega \quad .$$

The metric is obviously **SR-like** with $Q = 1/r$ and a proper length of $l = \ln(r)$, but it is not a vacuum as we defined before. The Einstein – and so the energy-momentum tensor has components $G_0^0 = \frac{1}{r^2}\frac{d}{dr}(r(g^{rr}-1)) = 3 - \frac{1}{r^2}$, $G_r^r = 1 - \frac{1}{r^2}$, $G_2^2 = G_3^3 = 1$ and for $r > 1$ the inequalities $G_0^0 \geq 0$, $G_0^0 \geq G_j^j \ \forall\, j = 1,2,3$, $G_0^0 > \sum G_j^j$ hold, which is the strong energy condition. So this ansatz leads to something else as what we expect and have defined as **vacuum.**

V.2 An "Almost-Schwarzschild" metric.

Now lets consider a solution of (IV.7) with $B_4=0$. To this set of solutions belongs the standard solution $B_1 \equiv B_r = m(r),\ B_j = 0, \forall\, j \neq 1$, which leads to the target metric as defined in section II , and there is no need to explain further. The next most simple, radial symmetric extension of this solution is $B_0 = \omega\,, B_1 \equiv B_r = m(r), B_2 = B_3 = 0$.
Now (IV.5) becomes, for $b=a$, as follows

$$ds^2 = (a^2-\omega^2)\,dt^2 - a^{-2}\,dr^2 - r^2\,d\,\Omega - 2\,\omega\, m\, dt\, dr \quad . \tag{V.13}$$

To get a Lorentz signature at least in some far away region, we demand $0 \leq \omega^2 < 1$. In order for the time parameter to remain still a cyclic coordinate (for simplicity), as in the usual theory, ω has to be constant. In this section we restrict ourselves to this case, and we will consider a non-constant ω in section VII . After redefining the time parameter $(1-\omega^2)\,dt^2 \rightarrow dt^2$ and using a redefined b^2 the metric is

$$ds^2 = b^2\,dt^2 - a^{-2}\,dr^2 - r^2\,d\,\Omega - 2\mathrm{m}\,\omega/\sqrt{(1-\omega^2)}\,dt\,dr\,,\ \ b^2 := (a^2-\omega^2)/(1-\omega^2) \quad . \tag{V.14}$$

The off diagonal term contains the square root of $m^2 = a^{-2} - (a' - a/r)^2$ (see (II.6)).
m^2 becomes negative in some shells inside r_0 around the origin (see II), and so the whole metric is complex. Now, with the help of the transformation
$d\,\tilde{t} = dt - (\omega/\sqrt{(1-\omega^2)} \cdot m/b^2)\,dr$, the metric diagonalizes. With $a^2 = 1+V$ (see (II.10)) and after renaming again $\tilde{t} \rightarrow t$ we receive

$$\begin{aligned} ds^2 &= b^2\,dt^2 - Q^2\frac{dr^2}{b^2} - r^2\,d\,\Omega\,,\ \ b^2 := 1 + \frac{V}{(1-\omega^2)}\,,\ \ Q^2 := \frac{1-\omega^2 f^2}{1-\omega^2} \\ f^2 &:= a^{-2} - m^2 = \frac{1}{(ar)^2}\Big(1+V-\frac{r}{2}\frac{dV}{dr}\Big)^2 \equiv (a/r - a')^2 \equiv r^2\Big(\frac{d}{dr}\frac{a}{r}\Big)^2 \end{aligned} \tag{V.15}$$

This metric is real everywhere, but has a few more singularities than the original target metric. The time scale factor b^2 of the metric has again the standard form $1+V$, but with a modified potential $V_h = \gamma_K V$.
As is obvious from the definition, (V.15) defines for $\omega=0 \Rightarrow b^2 = 1+V\,,\ Q^2=1$ the original "target" metric. For $V = -r_0/r + \lambda\, r^2$ (the Schwarzschild-de-Sitter metric) we have $f^2 = (1-3\mathrm{r}_0/2\mathrm{r})^2/(ar)^2$ and so $f^2(r\rightarrow\infty) = 0 \Rightarrow Q_\infty = 1/(1-\omega^2)$. The event horizon r_h: $b(r_h)=0$ is shifted to the point where $1+V(r_h) = \omega^2$. The qualitative main difference is/are the zero(s) of Q^2 . The radial part of the metric changes sign at those points and the proper radial distance l: $d^2 l = g_{rr}\,d^2 r$ has an extremum.
Let's consider in the following only the case $\lambda=0$, so we just have for $\omega=0$ the usual Schwarzschild-space (and the flat space) and the functions in (V.15) are

$$b^2 = 1 - \frac{r_h}{r}\,,\ r_h := \frac{r_0}{1-\omega^2} \quad \text{and} \quad f^2 = \frac{\kappa^2}{r_0^2}\frac{(1-3\,\kappa/2)^2}{(1-\kappa)}\,,\ \kappa := r_0/r \quad . \tag{V.16}$$

Now the metric already has the form of an **AS** metric, and $Q_\infty = (1-\omega^2)^{-1}$ to be constant also holds. We finally have to just calculate the Einstein tensor for $r_0 = r_h = 0$. So now we first consider the related **SR-like** space

-- the vacuum ($r_0 = r_h = 0$).

We look at the spatial line element (for any r_0) and rewrite the factor Q^2 as

$$Q^2 = 1 + \frac{\omega^2}{1-\omega^2}(1-f^2) =: 1 + k^2 g^2,\ k = \frac{\pm\omega}{\sqrt{(1-\omega^2)}}$$

and hence the proper length as $dl^2 = b^{-2}(1+k^2 g^2)dr^2$. Introducing, similarly as in [MT], a surface function $z: dz/dr = kg$ we get

$$dl^2 = b^{-2}(dr^2 + dz^2),\ \frac{dz}{dr} = \pm kg = \pm k\sqrt{(1-f^2)},\ k = \frac{\omega}{\sqrt{(1-\omega^2)}} \tag{V.17}$$

and for the complete line element

$$ds^2 = b^2 dt^2 - b^{-2}(dz^2 + dr^2) - r^2 d\Omega \quad .$$

This describes the 4-D space as an embedding in a 5-D space with an additional space-like dimension z .
For $r_0 = r_h = 0$ the function f is just $f = 1/r$ and so $dz/dr = \pm k\sqrt{(1-r^{-2})}$. The expression keeps only real for $|r| \geq 1$ and we found again the same restriction for "flat" spaces as we already got in section II (see (II.8) ff.) . As pointed out in II , this indicates that the space has a minimal resolution of one, which we identify as one Plank unit. For $r < 1$ the phase β of the KK - dimension gets an imaginary part, as now dz/dr becomes imaginary. We will discuss the line element closer below, but let's now finally calculate the Einstein tensor for $r_0 = r_h = 0$. With (V.3) we find

$$\begin{aligned} r^2 G_0^0 &= (1-\omega^2)\frac{d}{dr}\frac{r}{(1-(\omega/r)^2)} - 1 = -\frac{\omega^2(1-r^{-2})}{(1-(\omega/r)^2)} - \frac{\omega^2(1-\omega^2)}{r^2(1-(\omega/r)^2)^2} \\ &= -\omega^2\frac{(r^2+(\omega/r)^2+1-3\omega^2)}{r^2(1-(\omega/r)^2)^2},\quad G_0^0(r=1) = -2\frac{\omega^2}{(1-\omega^2)}. \end{aligned}$$

Because $r^2 + (\omega/r)^2 \geq 2\omega$ ($r^2 + (\omega/r)^2$ has a global minimum at $r^2 = \omega$), from $2\omega + 1 - 3\omega^2 \geq 0$ for $\omega \in [0,1]$ follows $G_0^0 \leq 0$, where the equal sign is only valid if $\omega = 0$. For the other components we get

$$r^2 G_1^1 = \frac{(1-\omega^2)}{(1-(\omega/r)^2)} - 1 = -\omega^2\frac{(1-r^{-2})}{(1-(\omega/r)^2)} \leq 0,\ \text{for}\, r \geq 1,\quad G_1^1(1) = 0 \quad \text{and}$$

$$G_2^2 = G_3^3 = \frac{-(1-\omega^2)}{2\mathrm{r}}\frac{d}{dr}(1-(\omega/r)^2)^{-1} = -\frac{\omega^2\cdot(1-\omega^2)}{r^4(1-(\omega/r)^2)^2} \leq 0.\quad G_2^2(1) = G_3^3(1) = \frac{-\omega^2}{(1-\omega^2)} \quad .$$

So the space is a **vacuum** in the strong sense for all $r \geq 1$. But even for $r < 1$ the calculation shows that the **vacuum energy-condition** $G_0^0 \leq 0,\ G_0^0 \leq G_j^j\ \forall j = 1,2,3$ holds and that the space is also a **vacuum** there as defined above.
At $r = 1$ the radial pressure G_1^1 is zero and changes sign crossing this value. The expression for G_0^0 simplifies slightly if $\omega^2 = 1/3$, and hence as a special case we get at $r = 1$ $G_0^0 = -1,\ G_1^1 = 0$, $G_2^2 = G_3^3 = -1/2$.

The energy density (its absolute value) decreases very rapidly with r^{-2} . The inner, non-asymptotic region, where r is small, has only an unmeasurable extension, if, as pointed out, the radial parameter is measured in plank units. The Bohr radius in these units, for example, is $r_{bohr}>10^{23}\Rightarrow f^2=1/r_{bohr}^2<10^{-46}$).
For an external stationary observer a gap in the range of r alone has no deeper meaning (see also the remarks after (V.11). He measures the proper length l and the time t , whereas both terms are the same, because of $dl/dt=1$ in a **vacuum**. So we have to analyze l . Integrating equation (VI.14) for(V.17) leads to

$$\begin{aligned} z&=\pm k(\sqrt{(r^2-1)}+\operatorname{asin}(1/r))+C \\ l&=\pm(\sqrt{(r^2-\omega^2)}+\omega\cdot\operatorname{asin}(\omega/r))/(\sqrt{(1-\omega^2)})+C \end{aligned} \qquad (V.18)$$

If we introduce the parametrization $r=-\omega/\sin(\xi),\ l=-k(\cot(\xi)-\xi),\ \xi\in[-\pi,0]$, we get for the proper length and hence also for the time a connected, strictly increasing curve ranging from $-\infty$ to ∞ and similarly for z . In parallel, for the values of the ξ - parameter, r comes from infinity, decreases until it reaches it's minimal value and returns back to infinity. Another possible parametrization with these features is $x=\omega\cosh(\xi),\ \ l=k(\sinh(\xi)+\operatorname{asin}(1/\cosh(\xi))),\ \ \xi\in[-\infty,+\infty]$. The second summand $\operatorname{asin}(1/\cosh(\xi))$ in the l-parametrization is the Gudermannian function $gd(\xi)=-\operatorname{asin}(1/\cosh(\xi))=\int_0^{\xi}\frac{dx}{\cosh(x)}$, "which arises in the inverse equations for the Mercator projection" (Wolfram.com). This may be a hint for someone to find an adequate geometry for the space. But in both parametrizations we do not get strict monotone increasing functions of r and $l=t$. A straight trajectory through the origin is just a one-dimensional line and the radial parameter must be exchangeable with a one-dimensional coordinate x ranging from $-\infty$ to ∞ . This we get with the following solution, discontinuous in x

$$\begin{aligned} z&=-k(\sqrt{(x^2-1)}+\operatorname{asin}(1/x)+\pi/2),\ \text{ for } x\in\ ]-\infty,-1] \\ z&=\ k(\sqrt{(x^2-1)}+\operatorname{asin}(1/x)-\pi/2),\ \text{ for } x\in\ ]1,\infty] \end{aligned} .$$

Now $z(1)=z(-1)=0$, so the trajectory starts at $z=x=-\infty$, reaches $x=-1,z=0$, than the x -value "jumps" to +*1* with $x=1,z=0$ and the trajectory goes on to $z=x=\infty$. Expressed with the inverse function, the r discontinuity would look like $r(z=-0)=-1,r(z=+0)=1$. For the proper length, the gap in the range of r is smaller. But remember the results in section II , where we found a minimal resolution of the radial dimension of $r\geq 1$. Below this value, the embedding is not possible for real phases β of the KK-dimension. So we introduce the same size of discontinuity in the relation $r(l)$ as for the z-dimension

$$\begin{aligned} l&=-\frac{\sqrt{(x^2-\omega^2)}+\operatorname{asin}(\omega/x)+\operatorname{asin}(\omega)}{\sqrt{(1-\omega^2)}}+1,\ \text{ for } x\in\ ]-\infty,-1] \\ l&=\ \frac{\sqrt{(x^2-\omega^2)}+\operatorname{asin}(\omega/x)-\operatorname{asin}(\omega)}{\sqrt{(1-\omega^2)}}-1,\ \text{ for } x\in\ ]1,\infty] \end{aligned} .$$

With this solution we get also that the Einstein-tensor fulfills the strong vacuum condition for all values of the proper length $l\in\mathbb{R}$. With this last parametrization, now even for $\omega=0$ the gap in the radial dimension exists. This may look at first unsatisfying, but on the other hand is consistent with the results in section II for minimal resolution.

"Far away" from the origin, in the Newton limit, the function becomes $r=\sqrt{(1-\omega^2)}l$. This relation between the two length systems $l^2=(1-\omega^2)^{-1}r^2$ in the Newton limit reflects just the Pythagorean identity $l^2=r^2+z^2=(1+k^2)r^2=(1-\omega^2)^{-1}r^2$. We may assume that we have $\omega\neq 0$ not everywhere and hence not everywhere this additional z-dimension and/or not any kind of matter/particles "sees" it (interacts with a related field). This picture is in some way similar (but as a visual picture only) to the drawing of a scalar electromagnetic potential over the radial distance as the z-dimension and using as path length again dr^2+dz^2 . The potential does not exist everywhere, and where it does exist, not just any kind of matter (only charged particles) is affected by it. Motivated by this, ω is some kind of interaction constant, so we call it a "charge". To "have not everywhere" $\omega\neq 0$ would imply that $\omega=\partial\beta/\partial t$ depends at least on r and therefore is not constant. Moreover, we then also get a time dependency for $m=\partial\beta/\partial r$ and after diagonalization (if possible) a general non-static metric. This leads to a more sophisticated and complex theory than described here. But even in the case of a non-constant ω , we assume that it is almost constant in large space areas and for large time spans, and our theory works in such regions.
Nevertheless, we have the general question, where ω comes from and which kind of matter it influences. In section VI we will provide some simple particle picture as an interpretation of our equations. This interpretation also suggests some kind of gravitation charge, with which the influence on the metric could switch on and off. We will point this out in more detail in section VI (and VII), while now we continue with the analysis of geodesics for the **AS** space for $r_0>0$

-- the extended Schwarzschild metric
As we have already seen above, in a geodesic (LeMaitre) coordinate system (V.12), anything looks quite familiar, but now we have to use the Kepler gravitation constant and so $r_h=r_0\gamma_K,\ \gamma_K=1/(1-\omega^2)$. For the circular geodesic equation (VI.11) the same picture holds, we just have to exchange the Schwarzschild radii. The metric acts as one with an extended mass compared to $r_0/2$.

For time-like geodesics, straight through the origin, we already have (V.11)

$$\dot{r}^2=\frac{r_h/r}{Q^2}=\frac{r_0/r}{1-\omega^2 f^2} \qquad \text{(V.19)}$$

with f^2 from (V.16) . For large r the denominator at the right-hand side of the equation becomes one, and so the equation becomes/is the standard one, as we also have for $\omega=0$. Finally, in the Newton limit with $dt\approx ds$, we end with Newton's law for a mass of size $r_0/2$ as usual for $\gamma_N=1$ and not for the "mass" $r_h/2$, which indeed is multiplied with γ_K . We already had generally discussed this effect in the text before (VI.10).
The embedding in section II , from which we deduced this metric, is defined $\forall r\geq r_0$. But now this space contains also an area $r_h\geq r\geq r_0$ lying behind the horizon r_h . Since f^2 ((V.16)) tends to infinity at r_0 , any value $\omega^2\neq 0$ generates some further singularities in (VI.19). Because $f(r=3/2r_0)=0$ and $\lim_{r\to\infty} f=\infty$, there exists at least one value r_f of the radial coordinate, $r_f:\ r_0<r_f<3/2r_0$, where $\omega^2 f^2(r_f)=1$ and hence $\dot{r}\to\infty,$ for $r\to r_f$. If $r_f<r_h$, the singularity at r_f is hidden behind the horizon. For a closer understanding of the trajectory we have to discuss f^2 , which shows:

- sign: $\omega^2 f^2 = \frac{\omega^2}{r_0^2}\kappa^2\frac{(1-3/2\kappa)^2}{(1-\kappa)} \geq 0$ for $0 \leq \kappa < 1$ $(r > r_0)$, $\omega^2 f^2 < 0$ for $\kappa > 1$ $(r < r_0)$,

- zeros: $f = 0$, for $\kappa =$ and for $\kappa = 2/3$,
- vertical asymptotes: $\lim_{\kappa\to\infty} f^2 = \lim_{\kappa\nearrow 1} f^2 = \infty$, $\lim_{\kappa\searrow 1} f^2 = -\infty$,
- derivative: $\frac{d(f^2)}{dr} = -\frac{\kappa^2}{r_0^3}\frac{d(f^2)}{d\kappa} = -(\kappa/r_0)^3\frac{(1-3/2\kappa)}{2(1-\kappa)^2}(9\kappa^2-14\kappa+4)$.

f^2 has a minimum at $\kappa_z = 2/3$, $r_z = 3/2 r_0$ (the minimum at zero corresponds to *r* at infinity and plays no role) and local maxima at $\kappa_{+/-} = \frac{(7\pm\sqrt{13})}{9}$ (and corresponding values $r_{+/-}$) and we have $0 < \kappa_- < \kappa_z < 1 < \kappa_+$, or $r_- > r_z > r_0 > r_+$.
As $f^2 < 0$ in $0 \leq r < r_0$, we have from (V.19) $\dot r^2 < \kappa$ in $0 \leq r < r_0$ and specially as $f^2 \sim 1/r^2$ in the origin $\dot r = 0$ at $r = 0$. This is in contrast to the usual theory, where $\dot r$ becomes unlimited there. Approaching r_0 from the left side f^2 tends to go again to infinity and we have again $\lim_{r\nearrow r_0} \dot r = 0$. The maximum at $r_+ < r_0$ plays no special role.
f^2 is strictly monotone for $\kappa \in \,]\kappa_z, 1[$ (or $r \in \,]r_0, \frac{3}{2} r_0[$) , and hence the right hand side of (V.19) has only one pole r_f (resp. $\kappa_f = r_0/r_f$) in this interval, and this pole is of order one. So $r_0 < r_f < r_z$ and for $r_0 < r < r_f$, the right hand side of (V.19) is negative. We may change the initial conditions and add in the numerator a constant $C = -r_0/r_f$, which fixes the sign and also resolves the pole. So we get also in the domain $r_0 < r \leq r_f$ a solution with finite $\dot r$ and $\dot r = 0$ for $r = 0$. This region is very special, in it the sign of g_{tt} and g_{rr} matches. If $r_h > r_f$ this region has an euclidean metric.
Approaching from the right side to r_f , the pole could not be resolved and $\dot r$ becomes infinite. If the value of $\omega^2 f^2$ at the maximum κ_- exceeds one, (V.19) contains two additional poles of the previous kind and analogous features at locations $r_p > r_z > r_f$.
Checking the value of $\omega^2 f^2$ leads to a critical ratio of $\omega^2 f^2(\kappa_-) \approx 0.23\frac{\omega^2}{r_0^2} < 1.$. For $r_0 \geq 1/2$, as we got in II, this inequality is always fulfilled. Moreover, if r_h is large enough, any of these poles would be hidden anyway.
Such that, <u>under all circumstances</u>, at least the first pole at r_f is hidden (i.e $r_h > r_f$), the event horizon has to be farther away from the origin, where the minimum of f^2 is located. This demand is expressed as $r_h \geq r_z = 3/2 r_0$ and so $\omega^2 \leq 1/3$ will have to hold. Since the critical ratio increases with ω , the best choice will be $r_h = r_z \Rightarrow \omega^2 = 1/3$. In this case also $\omega^2 f^2(\kappa_-) \ll 1$ holds for all $r_0 \geq 1/2$ and therefore no additional pole exist. At the event horizon $r = r_h$ (V.19) is again just $\dot r^2 = r_0/r$ as in the standard theory.
Summarizing we get the following picture of the geodesic. In "far" regions ($\kappa \to 0$) the equation of motion behaves as the usual one $\dot r^2 \approx r_0/r$, whereas for $r > r_h$ we have $\dot r^2 > r_0/r$. Crossing r_h , the radial dimension becomes time-like and vice versa, $\dot r$ increases to infinity at r_f . Behind r_f the trajectory speed is bounded and zero at the origin and at the "naked" Schwarzschild radius r_0 . To determine the sign of the acceleration near r_f we differentiate (V.19) to get

$$\ddot{r} = -\frac{r_0}{(1-\omega^2 f^2) r}\left(\frac{1}{r} - \frac{\omega^2}{1-\omega^2 f^2}\frac{df^2}{dr}\right) \approx -\frac{r_0}{\epsilon r_f}\left(\frac{1}{r_f} - \frac{\omega^2}{\epsilon}\frac{df^2}{dr}\right), \epsilon = 1-\omega^2 f^2 \quad .$$

Near r_f the derivative of f^2 is strictly positive (larger than some positive constant) and hence for $\epsilon \to \pm 0$ the acceleration is positive, that is the force is repulsive. Until now, we always implicitly assumed but never explicitly showed, that at least far away the forces are attractive. The last equation now shows this

$$\lim_{r\to\infty} \ddot{r} \approx -\frac{r_0}{r}\left(\frac{1}{r} - 2\frac{\omega^2}{r^3}\right) \quad .$$

So this metric has, additionally to what was said previously, the astonishing feature that from the "outside" it is attractive (and even for circular orbits it produces additional attractive forces), but it contains also an extreme repulsive shell around the origin.
Another remarkable feature of the metric is that it contains the region $r_0 < r \leq r_f$, where the sign of the time and the sign of the radial metric component match, as already mentioned. If, as assumed above, $r_h > r_f$, then the time and radial coordinate are space-like in this area and so the metric is euclidean, or as one says "the time is imaginary".
Finally we have to check the associated Einstein gravitation tensor. For $r_0 \neq 0$ one expects that at least in some regions it should fulfill some criteria of usual matter, that is the strong or at least the weak energy-condition [G,MT](as discussed for the vacuum).
Using with $\exp(\mu) = Q^2/b^2,\ \exp(\nu) = b^2 \Rightarrow \nu + \mu = \ln(Q^2)$ again the formulas in [HD 8.31 ff] , one gets from

$$r^2 G_t^t = (r(e^{-\mu} - 1))' = e^{-\mu}(1 - r\mu') - 1,\ r^2 G_r^r = e^{-\mu}(1 + r\nu') - 1$$

$$r \cdot \Delta := r \cdot (G_t^t - G_r^r) = -b^2/Q^2 \frac{d}{dr}\ln(Q^2) = +b^2/Q^4 \frac{\omega^2}{1-\omega^2}\frac{d}{dr} f^2 \quad .$$

Now the derivative of f^2 contains the factor $1 - 3/2\kappa$, which is equal to b^2 if $\omega^2 = 1/3$ and consequently the last factor $(9\kappa^2 - 14\kappa + 4) = (\kappa - \kappa_-)\cdot(\kappa - \kappa_+)$ of the derivative determines the sign of Δ , therefore $sign(\Delta) = -sign(\kappa - \kappa_-)\cdot(\kappa - \kappa_+)$. This sign is positive for $\kappa \in [\kappa_-, \kappa_+]$, which includes the region $r_0 \leq r \leq r_h$. For $\omega^2 \leq 1/3$, the local minimum $\kappa_m = 1/3 + (1-\omega^2)/2 \leq 1 - \omega^2 = r_0/r_h$ of the product $b^2(1 - 3/2\kappa)$ lies outside of $r_0/r_h \leq \kappa \leq 1 \Leftrightarrow r_0 \leq r \leq r_h$ and so this product is positive in $r_0 \leq r \leq r_h$.
Hence we have at least $\Delta \geq 0$ in $r_0 \leq r \leq r_h$ for all $\omega^2 \leq 1/3$ with $\Delta(r = r_h) = 0$. We would also arrive at this result for larger values of ω regions, where $\Delta \geq 0$, but for simplicity we will restrict ourselves in the following within this section to $\omega^2 \leq 1/3$.
Next we calculate for the angular components

$$G_2^2 = G_3^3 = e^{-\mu}\left(\frac{\nu''}{2} - \frac{\mu' - \nu'}{2}\left(\frac{1}{r} + \frac{\nu'}{2}\right)\right) = -e^{-\mu}\left(\frac{1}{r} + \frac{\nu'}{2}\right)\frac{(\nu+\mu)'}{2}$$

, where we used

$\frac{\nu''}{2} + \nu'\left(\frac{1}{r} + \frac{\nu'}{2}\right) = 0$ as for the usual Schwarzschild metric. With $\nu + \mu = \ln(Q^2)$ it

follows $G_2^2 = G_3^3 = \frac{b^2}{2r Q^4}\left(1 + \frac{r_h}{2rb^2}\right)\frac{\omega^2}{1-\omega^2}\frac{d}{dr} f^2 = \frac{\Delta}{2}\left(1 + \frac{r_h}{2rb^2}\right)$ and from this

$$G_0^0-G_1^1-G_2^2-G_3^3=-\Delta\frac{r_h}{2\mathrm{rb}^2}\geq 0 \text{ for } r_0\leq r\leq r_h$$, because $b^2\leq 0$ in this interval.

But also the factor $1+r_h/(2\mathrm{rb}^2)$ is positive for $r_0\leq r\leq r_h$ and $\omega^2\leq 1/3$ and hence G_2^2 and G_3^3 . At last we calculate the time component. We use the representation

$$-g_{rr}=\frac{1}{1-(r_h+\omega^2 q)/r}, q=(r-r_h)\frac{1-f^2}{1-\omega^2 f^2}$$ and derive

$$r^2 G_t^t=-(\frac{d}{dr}r\,g^{rr})-1=-\omega^2\frac{d}{dr}(r-r_h)q=-\omega^2(q+(r-r_h)\frac{d}{dr}q)$$

$$\Rightarrow\ r^2 G_t^t=-\omega^2(\frac{1-f^2}{1-\omega^2 f^2}-rb^2\frac{1-\omega^2}{(1-\omega^2 f^2)^2}\frac{d}{dr}f^2)=-\omega^2\frac{1-f^2}{1-\omega^2 f^2}+r^2\Delta\overset{!}{=}r^2(G_r^r+\Delta)$$

G_r^r is positive (or zero) in the interval, where f has values between 1 and ω^{-2} , $r\in[r_f,r_1],\ f(r_1)=1$, which is a subset of $[r_{0,}r_h]$. So at least in this region we have $G_0^0\geq G_j^j\geq 0\ \ \forall\ j=1,2,3,\ G_0^0\geq\sum G_j^j$ and so the strong energy-condition holds [G]. The null-energy $G_0^0\geq G_j^j\ \ \forall\ j=1,2,3$ condition already holds at least in $[r_{0,}r_h]$. The matter seems concentrated in a shell around the origin, which is located between the "naked" Schwarzschid radius r_0 and the event horizon r_h . On the interval border we have $f^2(r_0)=\infty,\ f^2(r_h)=0\ \Rightarrow\Delta(r_0)=\Delta(r_h)=0$ and hence

$$G_t^t(r_0)=-1/r_0^2,\ G_t^t(r_h)=-\omega^2/r_h^2=-\omega^2(1-\omega^2)/r_0^2$$

further $g_{tt}(r_0)=b^2(r_0)=\dfrac{-\omega^2}{(1-\omega^2)}\ \Rightarrow G_{tt}(r_0)=\dfrac{\omega^2}{(1-\omega^2)r_0^2}\overset{\omega^2=1/3}{=}\dfrac{1}{2\mathrm{r}_0^2}$ and so finally

$$G_{tt}(r=r_0=1)=1/2 \text{ and } G_t^t(r=r_0=1)=-1$$ for $\omega^2=1/3$.

Comparing this with the result for the vacuum at minimal coordinates (see above)

$$G_{tt}(r=1)=G_t^t(r=1)=-1$$,

shows again a similarity, which indicates, the special nature of the value $\omega^2=1/3$.

VI The non-static metric

In the last section we already mentioned the question, if ω is a global constant or not. Regardless if it is or not, the question arises, which kind of matter interacts with this charge,all or only some. If ω is not "global", it could depend on time or space or both. As already mentioned before, if ω is not constant, we arrive at a general non-static metric after diagonalization in any way. We may start again with equation (II.3), with arbitrary κ and solve the three PDE's in the three variables $t(\tau,r),\ \kappa(\tau,r),\ \beta(\tau,r)$ and for arbitrary prescribed functions $b(\tau,r),\ Q(\tau,r)$ resulting from $(r\kappa)^2 dt^2-d\kappa^2-d\beta^2=b^2 d\tau^2-(Q/b)^2 dr^2$. In this way we would get any kind of (radial symmetric) metric we want. So some further restrictions are needed to identify the, in some way "correct", metric. If one fixes $\kappa = r\,a$, one could investigate, for which kind of $b(\tau, r),\ Q(\tau, r)$ the PDE-system has a solution.

Lets start now again with the equation analogous to (V.13, with arbitrary $\omega = \partial\beta/\partial t\ , m = \partial\beta/\partial r$. Suppressing in the following whole section the radial term $r^2 d\Omega$, the line element is

$$ds^2 = (a^2 - \omega^2)\,dt^2 - (m^2 + f^2)\,dr^2 - 2\,\omega\, m\, dt\, dr\,,\quad f = d(a/r)/dr\,,\ a^2 = 1 - V \qquad \text{(VI.1)}$$

Now introduce coordinates $(\tau, \rho),$ where $t = t(\tau, \rho), r = \rho$, such that the metric diagonalizes and, after annihilating the off-diagonal terms, it gets the form

$$ds^2 = (a^2 - \omega^2)\,\dot{t}^2 d\tau^2 - (m^2 + f^2 + (a^2 - \omega^2)\,t'^2)\,d\rho^2,\quad \dot{x} := \partial x/\partial\tau,\quad x' := \partial x/\partial\rho \qquad \text{(VI.2)}$$

The condition that the off-diagonal terms cancels out is $p := t' = m\,\omega/(a^2 - \omega^2)$. Therefore, from the inter-convertibility of the second derivatives, we have

$$\partial\dot{t}/\partial\rho = \partial p/\partial\tau \quad (*).$$

Further, the equation $(\partial/\partial\tau)(\partial\beta/\partial\rho) = (\partial/\partial\rho)(\partial\beta/\partial\tau)$ must hold, and with $\partial\beta/\partial\tau = \partial\beta/\partial t \cdot \dot{t} = \omega\,\dot{t}$ and $\partial\beta/\partial\rho = (\partial\beta/\partial t)\cdot t' + \partial\beta/\partial r = \omega\, p + m = p\,a^2/\omega$ yields

$$(\partial/\partial\rho)(\omega\,\dot{t}) = (\partial/\partial\tau)(p\,a^2/\omega) = (\partial/\partial\tau)(q\,a^2),\quad q := p/\omega \quad (**).$$

To get a Schwarzschild-like time scale factor, we set $\dot{t} = 1/\sqrt{(1-\omega^2)} =: H\ \ (= \sqrt{(\gamma_k)}!)$. We would also be able to use the negative sign of the square root in this definition, because only $\dot{t}^2$ appears in the metric, but then we have with t and τ two opposite time directions. The one form $d\beta = \omega\,dt + m\,dr$ has in the new coordinate system the components

$$d\beta = \omega(\dot{t}\,d\tau + t'\,d\rho) + m\,d\rho = \omega\,H\,d\tau + a^2 q\,d\rho \quad . \qquad \text{(VI.3)}$$

<u>We rename for going forward the parameters</u> $\tau \to t,\ \rho \to r$ and use t, r instead of $\tau,\ \rho$.

Summarized, the PDE's (*) and (**) now take the form

$$\begin{aligned} \text{i)}\ \ \frac{\partial}{\partial r} H &= \frac{\partial}{\partial t} p \\ \text{ii)}\ \ \frac{\partial}{\partial r}\omega H &= \frac{\partial}{\partial t} q\,a^2 \quad , \end{aligned} \qquad \text{(VI.4)}$$

and with $a^2 = 1 + V$ the line element (without the radial part!) looks like

$$\begin{aligned} ds^2 &= b^2 dt^2 - \frac{Q^2}{b^2} dr^2,\ \ b^2 := 1 + H^2 V \equiv 1 + \gamma_K V\,, \\ Q^2 &= b^2 f^2 + m^2 a^2 \mathrm{H}^2 \end{aligned} \qquad \text{(VI.5)}$$

where p and q are

$$p = \omega H^2 m b^{-2}, \;\; q = H^2 m b^{-2} \;\; .$$

If we set $m^2 = a^{-2} - f^2$, we receive again, as in section V,

$$Q^2 = H^2 \cdot (1 - f^2 \omega^2) = (1 - f^2 \omega^2)/(1 - \omega^2) \quad \text{(VI.6)}$$

We can only make this selection, $m^2 = a^{-2} - f^2$, if ω is constant. In the general case, the PDE's VI.4 have to hold and restrict the form of m .
After multiplying the first equation of VI.4 with H and the second with ωH , both left sides of the equations become equal, due to $\partial/\partial r (H^2 (1 - \omega^2)) = 0$. Since a^2 is time-independent we get,

$$\frac{\partial}{\partial t} p = \omega \frac{\partial}{\partial t} a^2 q = a^2 \frac{p}{q} \frac{\partial}{\partial t} q \;\; \Rightarrow p = \omega_a(r) q^{a^2}, \;\; \omega = p/q = \omega_a(r) q^V \quad \text{(VI.7)}$$

with arbitrary function $\omega_a(r)$. In order to get a real ω correctly, we have to assume in those last equations that q is overall positive. But the general solution of the equation above over real functions would be $p = \omega_a(r) |q|^{a^2}$ and we have no restriction about the sign of the functions ω and q . This sign of ω and q will become an essential element of the solution. To simplify the calculation for any sign, we use VI.7 , but allow, for the moment, complex values for ω and q . Finally we will end again with just real functions. So for $V = 0 \Rightarrow a^2 = 1$ we get already the solution

$$p = \omega_a(r) q \Rightarrow p/q \equiv \omega = \omega_a(r) \;\; .$$

This solution does not depend on t , which means that the characteristics, belonging to the PDE's VI.4 , are just the spheres of constant radius, and the initial values $\omega_0 = \omega_a$ are already the solution for all t , $\omega(t, r) = \omega(0, r) = \omega_0(r) = \omega_a(r)$ (for the theory of characteristic equations of PDE'S and associated initial value problem see any mathematical text book about first order PDE'S e.g. [Bi] we will point this out in more detailed below). Without any ambiguity therefore ω_0 denotes the initial values of ω and the solution for $V = 0$.

From the second equation of VI.4 then for $V = 0$ and with

$$\frac{\partial H}{\partial \omega} = H^3 \omega \;, \; \frac{\partial \omega H}{\partial \omega} = H^3 \;\; \text{follows}$$

$$q = H^3 \omega_{,r} \cdot t + \tilde{m}(r) = H^2 m \;\; \Rightarrow m = H \omega_{,r} \cdot t + m_c(r), \;\; \tilde{m} = H^2 m_c \quad \text{(VI.8)}$$

with some arbitrary function $\tilde{m}(r)$ or $m_c(r)$, respectively. To receive also the results of section V for the metric with constant ω , we have to set $m_c^2 = 1/a^2 - f^2$, which is in the current case, $V = 0$, just $m_c^2 = 1 - r^{-2}$. Note, we can use both signs ± 1 for m_c without changing the resulting metric. For completeness, we explicitly use VI.4 to calculate $\beta = \omega H t + g(r), \;\; dg/dr = \tilde{m}$ (please note that we renamed the coordinates).
Generally we have from VI.7 and $V = -r_0/r, \; r_0 > 0$ (our common use for V) the relations

$$\omega = \omega_a q^V, \;\; \omega = p^{V/a^2} \omega_a^{1/a^2} = p^{-r_0/(r - r_0)} \omega_a^{r/(r - r_0)}, \;\; p = \omega_a^{r/r_0} \omega^{-(r - r_0)/r_0} \;\; . \quad \text{(VI.9)}$$

Equation VI.9 allows us to view ω as function of r and of $p(r,t)$. H then is of type $H(p(r,t),r)$ and the partial derivative on the left side of VI.4 transforms to $\partial/\partial r \rightarrow \partial/\partial r+(\partial p/\partial r)\cdot\partial/\partial r$ [5] and so the first equation becomes

$$H_{,p}\cdot p_{,r}+H_{,r} \equiv H_{,\omega}\cdot(\omega_{,p}\cdot p_{,r}+\omega_{,r}) = \frac{\partial}{\partial t} p \quad . \tag{VI.10}$$

Now we transform it into the equivalent ODE system of characteristic equations

$$\frac{dp}{ds} = \frac{\partial p}{\partial s}\cdot\frac{dt}{ds}+\frac{\partial p}{\partial r}\cdot\frac{dr}{ds} = H_{,r}\ ,\ \frac{dr}{ds}=-H_{,p}\ ,\ \frac{dt}{ds}=1 \quad .$$

Because of $dt/ds=1$, we can also use the time t as parameter along the characteristics and end up with the canonical Hamilton-Jakobi equations

$$\frac{dr}{dt}=-H_{,p},\ \frac{dp}{dt} = H_{,r} \quad . \tag{VI.11}$$

Any solution of the PDE VI.10 satisfies VI.11 , and from a solution of VI.11 one constructs, via resolving and insertion, solutions of VI.10 (we will do this explicitly below). The Hamilton-Jakobi equations VI.11 have the opposite sign as usual. We could change some definitions to get the convenient sign. The first option, to change the definition of $\dot{t}$, was already discarded in the beginning, because we would get time in the opposite direction. But we can also redefine $p \rightarrow -p$ or $H \rightarrow -H$ to get the convenient form. If VI.11 is interpreted as the equations for classical particle path, then one may assume, that we have to deal with a negative energy $-H$. We delay this discussion about the interpretation and correspondence with particle paths to the end of this section. That H , and equivalently ω , is indeed "conserved" along the characteristic curves, we know (or immediately deduce) from VI.11 ,

$$\frac{dH}{dt} = \frac{\partial H}{\partial p}\cdot\frac{dp}{dt}+\frac{\partial H}{\partial r}\cdot\frac{dr}{dt} = 0 \quad \text{and so}$$

$$\frac{d\,\omega}{dt} = 0 = \frac{\partial\omega}{\partial t} + \frac{\partial\omega}{\partial r}\cdot\frac{dr}{dt} = 0 \ \Rightarrow\ \omega_{,t}=-\frac{dr}{dt}\cdot\omega_{,r} \quad . \tag{VI.12}$$

From VI.9 we know that the factor ω_q is already determined by the initial values,

$$\begin{aligned} &\omega_a(r) = \omega(t,r)q(t,r)^{-V} = \omega(0,r)q(0,r)^{-V} =: \omega_0(r)q_0(r)^{-V} \\ &\Rightarrow \qquad q = q_0(r)\cdot(\frac{\omega}{\omega_0})^{1/V} = q_0(r)\cdot(\frac{\omega_0}{\omega})^{r/r_0} \quad . \end{aligned} \tag{VI.13}$$

Again we do not care about the sign. Similar to what we already said following VI.7 , the correct equation would contain the absolute values of the ratio $\omega(t,r)/\omega_0(r)$, and q does not become complex-valued. q and q_0 carry always the same sign and have the same zeros. We can't deduce that ω shares the sign of its initial value at any coordinate r , and will be a special feature of thes olution. But because ω is conserved, it will carry the sign of ω_0 , if the initial values ω_0 never change their sign.

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5 In the sense of $\partial_r H(t,r) \rightarrow \partial_2 H(p,r)+(\partial_1 H(p,r))(\partial_2 p(t,r))$

Now we use VI.9 to calculate

$$\frac{\partial H}{\partial p} = H^3 \cdot \omega \frac{\partial \omega}{\partial p}, \quad \frac{\partial \omega}{\partial p} = \frac{V}{a^2} \cdot \frac{\omega}{p} = -\frac{r_0/r}{(1-r_0/r)} \cdot \frac{1}{q} \quad (=0 \text{ if } V=0)$$

and then, substituting it in VI.11 ii.)

$$\begin{aligned} \frac{dr}{dt} &= -H_{,p} = -H^3 \omega \cdot \frac{\partial \omega}{\partial p} = -H^3 \omega \cdot \frac{V}{a^2} \cdot \frac{1}{q} \Rightarrow \\ q \cdot (r-r_0) \frac{dr}{dt} &= r_0 H^3 \cdot \omega, \text{ or } \mathrm{v} := \frac{dr}{dt} = \frac{r_0}{r} \frac{H^3 \cdot \omega}{a^2 q} \end{aligned} \tag{VI.14}$$

For $r_0=0$ ee get the result, as we already mentioned, that the characteristics are the spheres of constant radius $r=r_A$. To determine, via the characteristic equations, the whole solution, we have to also solve the first Hamilton-Jakobi equation to get, with the abbreviation $G:=H^3 \cdot \omega \cdot \omega_{,r}$, $p=p_A+G(r_A)t$. Using the first result $r=r_A$ then leads to $p=p_A+G(r)t \Rightarrow \pi(p,r,t):=p_A=p-G(r)t$. From general theory (see e.g. [Bi]) one gets the set of all possible solutions. Specifically, any solution of the equation $\pi(r,t)-h(r)=const.$ also solves the PDE (the left side contains conserved quantities along the characteristic, and so the total derivative vanishes and could be compared with the partial derivatives of the equation, see [Bi]). This ends in the result $p=h(r)+G(r)t$ or equivalently VI.8 .

In the case of $V\neq 0$ equation VI.13 already results in, from initial values for ω and p or q and the solution for ω , the solution for p and q . To get a smooth solution for all time, we need non-crossing characteristics, or in the terminology of [CF] rarefaction waves. One may assume that one could perhaps also get weak solutions (in the sense of [L], which are shock waves (see [CF,L]), i.e. with "crossing" characteristics. In our case, the jump conditions [CF,L] over the shock front of VI.4 are just $[H]=0$ and $[\omega H]=0$, where the bracket [.] denotes the difference of the value inside over the discontinuity. So ω has to be continuous in any case, and we need, if we want to have a solution for non constant ω for all t,r , non- crossing, i.e parallel, characteristics. This condition is fulfilled, as we have seen for $r_0=0$, but could not be fulfilled for $r_0\neq 0$. We may be able to prescribe initial values, that the characteristics diverge in the forward time or in the backward time direction, but not in both. This may look at first unsatisfying, because we will not get a metric for all time t . But a model with constant mass $r_0\neq 0$ could in any way only be valid for a certain time span. No isolated massive object, neither in the macroscopic world nor in the microcosms (expect perhaps electrons), remains constant or has existed and will exist for all time. Moreover, one may interpret this non-existence of a solution for all t that any massive particles have at least a time point, where they are created or where they are destroyed (decayed). Combining now VI.13 and VI.14 yields

$$q_0(r)\left(\frac{\omega_0(r)}{\omega}\right)^{-1/V}(r-r_0)\frac{dr}{dt} \;=\; r_0 H^3 \omega \quad . \tag{VI.15}$$

This equation is principally integrable with a solution that goes through $t=0, r=r_A$ as

$$R(r,\omega)-R(r_A,\omega) \;=\; r_0 H^3 \omega t \quad . \tag{VI.16}$$

To get the solution for ω , one has to now resolve the last equation for r_A , which results in some function $r_A = \rho(t, r, \omega)$. Via the implicit function theorem we will then arrive at a solution $\omega(t, r)$ in some neighborhood of $t=0$ by solving the equation $\omega = \omega_0(\rho)$ and hence we will get , at $t=0$ the initial values $\omega_0(r)$ because of $\rho(0, r, \omega) = r$. To be more precise, to be able to conclude this without restrictions, $((r - r_0) q)^{-1}$ has to be Lifschitz-continous. We postpone discussing the related problematic.
If the initial values $\omega = \omega_0(r)$ are injective and hence invertible for all r with $r = \eta(\omega), \ \eta \circ \omega_0 = 1$, we can immediately insert this into VI.16 and get

$$R(r, \omega) - R(\eta(\omega), \omega) \ = \ r_0 H^3 \omega t \qquad \text{(VI.17)}$$

which then again defines via implicit function theorem the solution $\omega(t, r)$ in some neighborhood of $t=0$, in an alternative manner to before.
To get a first idea for the shape of the characteristics, we assume $q \approx q_0$ and start with an approximation of VI.15 ,

$$q_0 \cdot (r - r_0) \frac{dr}{dt} \ = \ r_0 H^3 \omega \ .$$

For $|q_0| \approx 1$ integration leads to the parabolas $(r - r_0)^2 = (r_A - r_0)^2 \pm H^3 \omega t$.

We continue with a
<u>- remark about initial values</u>.
If we want now prescribe the same initial values q_0 as in the case of $V=0$ before (or for constant ω in section V), we have to set again

$$q_0 = m_0 / (a^2 - \omega_0^2) \ \text{ with } \ m_0 = \pm m_c, \ m_c = +\sqrt{a^{-2} - f^2} = a^{-1} \sqrt{1 - (ar \frac{d}{dr} \frac{a}{r})^2} =: a^{-1} \hat{m}_c \ .$$

As we already mentioned, we can use both signs of the square root without changing the metric. So we have $(a^2 q_0)^{-1} = H^{-2} \frac{b^2}{a} \hat{m}_c^{-1}$, which is zero at $r = r_h$ and smooth and bounded for all $r \geq r_h$. If $\lim_{r \to \infty} |(\omega_0 / \omega)| > 1$ the factor $(\omega / \omega_0)^{(r/r_0)}$ would go to infinity and so $\dot{r}$ and we will at least not get solution of the initial value problem in this limit. This we have to in mind in all of the following. But now we give another

-**approximate solution for small** r_0 / r .

For this we have first a look at the general solution VI.8 for $r_0 = 0$ in the form

$$\omega = \omega_0(r), \ q = H^3 \omega_{,r} \cdot t + q_0(r) \ .$$

If this is also an approximation of the exact solution for $r_0 \neq 0$ with the same initial values, via VI.13 ,

$$(q / q_0)^{-r_0/r} = (1 + H^3 \omega_{,r} \cdot t / q_0(r))^{-r_0/r} = \omega / \omega_0 \ \text{ has to hold.}$$

Assuming that

$$\omega_{,r} \approx \omega_{0,r}, \ \left| t H^3 \frac{\omega_{0,r}}{q_0} \right| \ll 1 \ \Rightarrow \log(1 + \frac{t H^3 \omega_r}{q_0}) \approx \frac{t H^3 \omega_{0,r}}{q_0} \ \text{ we receive}$$

$$\omega \approx \omega_0(r)\cdot\exp\left(\frac{-r_0}{r}\frac{H^3\omega_{0,r}t}{q_0}\right) \approx \frac{\omega_0}{\left(1+\frac{r_0}{r}\frac{H^3\omega_{0,r}t}{q_0}\right)} \quad . \tag{VI.18}$$

Before we sketch in which way VI.18 is indeed an approximation of the exact solution, we must have a critical look at the assumptions we made. Because of the boundedness of ω_0 , $0\le\omega_0^2\le 1$, we can conclude that its derivative tends to zero for large r . But we need first the stronger assumption that $H^3|\omega_0'|=\left|\frac{d}{dr}(H(\omega_0)\cdot\omega_0)\right|$ tends to zero for large r . Then assuming q_0 is of size 1 for large r , as e.g. the q_0 which we used above, we get for fixed t and large enough r indeed $H^3|t\,\omega_0'|\ll|q_0|$. However this precondition makes no explicit use of the ratio r_0/r and so we not can conclude, that in generally VI.18 could be a an approximation of first order in r_0/r , even it looks like. One way to introduce this ratio in the inequality is to demand, that ω_0 depends on r_0 with $\omega_0' = O(r_0/r)$. This assumption is not unjustified, if we think, that this initial values are the result of some previous state with $r_0\neq 0$. But if we assume this for every allowed initial values, we would get for $r_0=0$ only the trivial case with ω_0 is a constant. But, using $r_0\ge 1/2$ from section II , we may argue with $1/r\le 2\mathrm{r}_0/r\ll 1$, that a small ratio r_0/r implies automatically a large r . Without any of this two arguments, an exact solution for $r_0\neq 0$ will NOT tend for $r_0\to 0$ to the solution with $r_0=0$!

One main difficult in VI.14 is the singularity at $a^2=0$. At it, the velocity v becomes infinity and it's sign may change. Here we want to consider the case of small r/r_0 and not go into this problem. Therefore and that our assumptions not contradict, we have to demand, that the characteristics are directed outside, which is $\mathrm{v}\ge 0$ if we consider solutions for $t\ge 0$ or $\mathrm{v}\le 0$ for $t\le 0$. We handle the case $t\ge 0$, the second one could be treated analogous.

To continue now, we write VI.16 in the form

$$\int_{r_A}^{r} dx((x-r_0)\cdot q(x,\omega)) \;=\; t\cdot H^3(\omega)\cdot\omega \quad . \tag{VI.19}$$

Note that ω is a constant with respect to the integration parameter x , but depends implicitly (after inserting the final solution) on the upper integration limit r and t . Applying the mean value theorem we get

$$q(\hat{r})(\hat{r}-r_0)\cdot(r-r_A)=r_0 t\,H^3\omega\,,\;\; \hat{r}\in[r_A,r] \quad . \tag{VI.20}$$

Now $\mathrm{v}\ge 0$ and $t\ge 0$ implies $sign(q_0)=sign(\omega)$ (we prescribe q_0 and use a proper constant ω in this way) . Assuming q_0 is bounded away from zero, i.e. for some $r_1>r_0$ and constant c_0 $\inf_{r\ge r_1}|q_0|\ge c_0>0$ holds, we need also $sign(\omega_0)=sign(\omega)$, i.e. ω_0 does not change sign, and $\inf_{r\ge r_2}\omega_0/\omega\ge 1$ for some $r_2>r_0$. From that we get $sign(\omega)\cdot q(r)=|q_0(r)|(\omega_0(r)/\omega)^{r/r_0}\ge c_0$ for $r\ge r_3=max(r_1,r_2)$ and so with VI.20 at least for $r_A\ge r_3$ $(\hat{r}-r_0)\cdot(r-r_A)\le c_0^{-1}r_0 t\,H^3|\omega|$. Since also $\hat{r}>r_0$ and the right hand side is positive and not depends from r , $\delta(r):=r-r_A$ has to be positive and bounded for $r\to\infty$. Accordingly $r_A = r-\delta(r)$ tends also to infinity for $r\to\infty$ and hence also $\hat{r}\to\infty$. So asymptotically $0\le\delta(r)<c_1 r_0/r$, i.e. $\delta(r)=O(r_0/r)$, holds and in the same way $r-\hat{r}=O(r_0/r)$. Knowing this, we use it recursive to determine $\delta(r)$ more precise. For q_0 sufficiently smooth we write now VI.20 as

$$\delta(r)=\frac{r_0}{r}\frac{H^3\omega t}{(\hat{r}/r-r_0/r)\cdot q(\hat{r})}=\frac{r_0}{r}\left(\frac{H^3\omega t}{q_0(r)}\cdot\frac{1}{X(\hat{r})}+O(r_0/r)\right),\quad X(r):=\frac{q(r)}{q_0(r)} \quad .$$

Now we need again that r is large, and not only that r_{0r}/r is small, and additionally that $\omega_0{}'$ tends to zero for large r , to be able to conclude

$$\begin{aligned} X(\hat{r}) &= \left(\frac{\omega_0(\hat{r})}{\omega}\right)^{\hat{r}/r_0}\approx\left(\frac{\omega_0(\hat{r})}{\omega}\right)^{O(r_0/r)/r_0}\cdot\left(\frac{\omega_0(r)+\omega_0{}'O(r_0/r)}{\omega}\right)^{r/r_0}\\ &\approx\left(\frac{\omega_0(r)}{\omega}\right)^{r/r_0}\left(1+\frac{r}{r_0}\frac{\omega_0{}'}{\omega}O\left(\frac{r_o}{r}\right)\right)\approx\left(\frac{\omega_0(r)}{\omega}\right)^{r/r_0}=X(r)\end{aligned}$$

and get

$$\delta(r)\approx\frac{r_0}{r}\frac{H^3\omega t}{q_0(r)}\cdot\frac{1}{X(r)} \quad \text{for large enough } r \ .$$

Expanding the equation $\omega=\omega_0(r_A)$ becomes $\omega=\omega_0(r)+\omega'_0(r)\delta(r)+O(\delta(r)^2)$ and so in first order

$$\omega(r)=\omega_0(r)-\omega_{0,r}(r)\frac{r_0}{r}\frac{H^3\omega t}{q_0(r)}\frac{1}{X}\;\Rightarrow\;\left(1+\frac{r_0}{r}\frac{H^3\omega_{0,r}t}{q_0}\frac{1}{X}\right)=\frac{\omega_0}{\omega}=X^{r_0/r}$$

. In the same order this now implies

$$\omega(r)=\omega_0(r)-\omega_{0,r}(r)\frac{r_0}{r}\frac{H^3\omega t}{q_0(r)}\frac{1}{X}\;\Rightarrow\;\left(1+\frac{r_0}{r}\frac{H^3\omega_{0,r}t}{q_0}\frac{1}{X}\right)=\frac{\omega_0}{\omega}=X^{r_0/r}$$

$$\left(1+\frac{r_0}{r}\frac{H^3\omega_{0,r}t}{q_0}\frac{1}{X}\right)=1+\frac{r}{r_0}\ln(X)\Rightarrow X\ln(X)=\frac{H^3\omega_{0,r}t}{q_0} \quad .$$

With the additional demand from above $tH^3|\omega_0{}'/q_0|\ll 1$ the last equation has the approximate solution $X=1+tH^3\dfrac{\omega_{0,r}}{q_0}\;\Rightarrow q=q_0+tH^3\omega_{0,r}$, which is the solution of q in the case of $r_0=0$.

Summarized we have:

If ω_0 not change sign and $H^3|\omega_0{}'|=\left|\dfrac{d}{dr}(H(\omega_0)\cdot\omega_0)\right|$ tends to zero for large r and q_0 is bounded away from zero, the solution for $r_0\neq 0$ of the initial value problem tends for large r to the corresponding solution for $r_0\neq 0$ and VI.18 holds. For a smooth monotone and bounded ω_0 , all its derivatives will tend to zero for large r and the same holds for q_0 . So VI.18 induces, that ω tends absolutely with all its derivatives to ω_0 .

From a physical point of view, something which modifies the metric of space-time, should be correspond with some kind of particle/matter. So we now give

-**a simple particle picture of the equations**

For this we first reformulate the Hamilton-Jakobi equations. We had derived them using the first equation of VI.4. But we can use VI.4 ii.) in the same way. The result VI.14 , of course, will remain the same, but the structure of the equations will give new ideas for it's interpretation. Instead of (p,r) we now use (P,r) , $P:=-a^2 q$, as canonical coordinates and with $E:=H\omega$ derive, analogous to VI.11,

$$\frac{dr}{dt}=-E_{,\mu}\,,\ \ \frac{dP}{dt}=E_{,r} \qquad \text{(VI.21)}$$

Due to the form of these equations we already may identify P as the momentum and E as the energy of the searched particle . E is obviously also "conserved". Moreover with these definitions the equation VI.3 has the form (attention we renamed $\tau\rightarrow t,\ \rho\rightarrow r$)

$$d\beta = E\,dt - P\,dr \quad .$$

Originally we introduced β as the phase of the Kaluza-Klein dimension (see II.4) . In complex notation (as seen in section III) the Kaluza-Klein dimension is written as $z=e^{i\beta}$ and from this we get

$$i\frac{\partial\bar{z}}{\partial t}=E\,\bar{z}\,,\ \ -i\frac{\partial\bar{z}}{\partial r}=P\,\bar{z} \quad , \qquad \text{(VI.22)}$$

which already gives us the announced particle interpretation.

For consistence, the characteristics VI.14 should determine the path of this particle, and so its momentum has to point in the direction of the velocity $v:=dr/dt$. Therefore we demand $sign(v)=sign(P)=-sign(q)$, which results with $sign(v)=sign(\omega\cdot q)$ (from VI.14) in $sign(\omega)=-1$ or $E\leq 0$.
A negative energy would indicate that this particle is some sort of anti-matter. But the fact that the direction of the momentum and velocity coincide means that it moves "forward in time" and therefore is some kind of "ordinary matter". Its anti-particle, which moves "backward in time" (its momentum anti-parallel to the velocity), in contrast has positive energy. So then our usual description of matter lacks, like it does for the "exotic matter" mentioned in section V, page 17. That our particle could not be some common kind of matter or anti-matter also follows from the observation, that it DOES NOT travel on geodesics of space-time, instead its trajectories are solutions of VI.14. Moreover, the equations describe the interaction of these particles with matter/anti-matter (and also the vacuum), but not amongst each other, and to assign some mass to this particle itself seems at least problematic. So we still do not use the term matter, especially not "exotic matter". As for other particles, we have for our Њ-particle (to give it some name), which has $E<0$, a corresponding anti-Њ-particle with $E>0$.
With the current definitions we may rewrite VI.14 as

$$P\,v=-\frac{r_h}{r}E \quad , \qquad \text{(VI.23)}$$

which shows an astonishing similarity to the analogical equation $P\,v=2E$ for a free particle in classical mechanics. But while in mechanics $v=0$ implies $E=0$, in our case it could be also mean $r_0=0$. Note, the velocity v defined here is not the velocity a static observer measures. For this we have to multiply by the factor Q (see V.7) and

get $v_{SO}^2=(vQ)^2=(\frac{r_h}{r})^2(\frac{\omega}{a})^2(1+\frac{(a^2-\omega^2)}{(am)^2}f^2)$). It tends to zero for large r and has the value $v_{SO}^2=1$ at $r=r_h\Rightarrow\omega^2=a^2$.
While the energy E is conserved, the momentum P is only conserved if E is a global constant. This does not contradict physical laws, because if E is not equally distributed (not constant), therefore the space not isotropic and momentum conservation could not be deduced.
Since for usual matter and anti-matter gravitational interaction is attractive, we assume, because of the opposite sign of the energy, a repulsive character of forces arising with Њ-particles. This in turn suggests the further assumption that we always must have $v\geq 0$ in VI.23 . So the particle path starts at r_h with velocity 1 and ends at infinity with speed 0 . The velocity is identically zero for $r_0=0$, which means that only usual matter and anti-matter act repulsive on Њ-particles and anti-Њ-particles, but that they do not interact with each other. From $v\geq 0$ i.e. $sign(v)=1$ we get

$$sign(q)=sign(\omega)=sign(E) \quad . \tag{VI.24}$$

In the following, we demand again that the initial values have ~~no zero and so~~ a unique positive or negative sign. We consider two distinct assumptions about the absolute value of the particle energy, I) $\frac{\partial E^2}{\partial t}\leq 0$ and II) $\frac{\partial E^2}{\partial t}\geq 0$. Further with $\frac{\partial E^2}{\partial t}=H^4\frac{\partial\omega^2}{\partial t}$ and VI.12 ($\omega_{,t}=-v\omega_{,r}$) follows

$$\begin{array}{l} \text{i.)}\ \frac{\partial\omega^2}{\partial t}\leq 0,\ \frac{\partial\omega^2}{\partial r}\geq 0 \\ \text{ii.)}\ \frac{\partial\omega^2}{\partial t}\geq 0,\ \frac{\partial\omega^2}{\partial r}\leq 0 \end{array} \quad . \tag{VI.25}$$

As a combination of VI.24 + VI.25 we also have

$$\begin{array}{l} \text{i.)}\ q\omega_{,r}\geq 0 \quad \text{and equivalently} \quad qE_{,r}\geq 0 \ , \\ \text{ii.)}\ q\omega_{,r}\leq 0 \quad \text{and equivalently} \quad qE_{,r}\leq 0 \ . \end{array} \tag{VI.26}$$

We are able to prescribe initial values such that VI.24 + VI.25 hold, but for which time interval these features remain valid, we have to derive from the solution. Note, applied to initial values, the second equation in VI.25 again contains the problem we already had for deducing the solution for small r_0 . For any ω , with $\omega^2>\lim_{r\to\infty}\omega_0^2(r)$, q will tend to zero and so the velocity v to infinity. But in spite of this, in this section we also will treat this second case.
Assuming initial values, which do not change sign and for which VI.24 holds, than this equation will hold in the whole time interval where the solution exists. The solution is given implicitly (see above) as $\omega=\omega_0(\rho_t)$, with ρ_t defined via solving VI.16 $r_A=\rho_t(r,\omega)$ for a fixed time t . Now, because of VI.24 , the initial values are injective. If we got another injective function $\omega_t(r)$, which solved the PDE, we would be able to connect both solutions uniquely via $r_t=\omega_t^{-1}\circ\omega_0(r_A)$. If we can do this for all $t':t\geq t'>0$ (or $t':t\leq t'<0$ if *t* is negative) we have a unique solution in the whole interval, with the prescribed initial values. First we have

$$\omega_{t,r}=\omega'_0(\rho_t)\rho_{t,r} \quad . \tag{VI.27}$$

To calculate the $\rho_{t,r}$ we start again with VI.19 (with slightly changed nomenclatura)

$$\int_{\rho_t}^{r} dx(x\cdot q(x,\omega_t)) \ = \ t\cdot r_h(\omega_t)\cdot E(\omega_t)$$

Note again that ω_t is a constant with respect to the integration parameter x , but after inserting the solution, it depends on the upper integration limit r , as ρ_t does. Further for any solution of the equation we must have

$$sign(r-\rho_t)\,sign(q)=sign(t)\,sign(\omega) \tag{VI.28}$$

We differentiate the function with respect to r and receive

$$\int_{\rho_t}^{r} dx(x\cdot q_{,\omega}(x,\omega_t)\omega_{t,r})+r\cdot q(r,\omega_t)-\rho_t\cdot q(\rho_t,\omega_t)\rho_{t,r} \ = \ t\cdot r_h(\omega_t)\cdot E(\omega_t) \quad ,$$

where $G=\partial(r_h E)/\partial\omega_t>0$, and after applying the mean value theorem

$$\hat{r}\cdot q_{,\omega}(\hat{r},\omega_t)\omega_{t,r}(r-\rho_t)+r\cdot q(r,\omega_t)-\rho_t\cdot q(\rho_t,\omega_t)\rho_{t,r} \ = \ t\cdot G(\omega_t)\omega_{t,r}$$

with $\hat{r}\in[\rho_t,r]$. Because $q_{,\omega}=-r/r_0\cdot q/\omega$ we find

$$\rho_{t,r}=\frac{r\cdot q(r,\omega_t)}{\rho_t\cdot q(\rho_t,\omega_t)}-\frac{\dfrac{\hat{r}^2}{r_0}\dfrac{q(\hat{r},\omega_t)}{\omega_t}\cdot(r-\rho_t)+t\cdot G(\omega_t)}{\rho_t\cdot q(\rho_t,\omega_t)}\omega_{t,r} \quad .$$

The first summand on the right hand side is positive, and the ~~nominatorde~~numerator of the second summand has, because of VI.28 , the sign of t . Insertion into VI.27 leads hence to an equation of the form
$\omega_{t,r}(1+\omega_0' A)=\omega'_0 B$, with $sign(A)=sign(q)\,sign(t)$ and positive B .
Because of $sign(\omega_0')\,sign(q)=sign(\omega_0'\omega)=sign(\omega_0'\omega_0)$ we have

$$sign(\omega_0' A)=sign(t\cdot\omega_0'\omega_0)$$

and so for $sign(t\cdot\omega_0'\omega_0)\geq 0$ $\quad sign(\omega_{t,r})=sign(\omega'_0)$.

So we proved that with initial values $\dfrac{\partial E^2}{\partial t}\leq 0$ (or $\dfrac{\partial E^2}{\partial t}\geq 0$, resp.) for all $t\geq 0$ (or $t\leq 0$, resp.) the radial derivative inherits the sign of the derivative of the initial values. Consequently a solution exists for all $t>0$ with these initial values and VI.25 i.) (or ii.) + VI.26i.) (or ii.) holds.

We remark also: In order for ω^2 to monotonically increase/decrease and be bounded $0\leq\omega^2\leq 1$, its tangent has to decrease at least asymptotically, i.e for large r we must have

$$\text{i.)}\ \frac{\partial^2\omega^2}{\partial r^2}\le 0,\ \ \text{ii.)}\ \frac{\partial^2\omega^2}{\partial r^2}\ge 0\quad\text{and}\quad \lim_{r\to\infty}\frac{\partial^2\omega^2}{\partial r^2}=0 \tag{VI.29}$$

for a "smooth enough" convergence.

Using VI.18 we get also an approximation for the time development

$$\frac{\partial\omega^2}{\partial t}=-2\frac{r_0}{r}\frac{H_0^3\omega_{0,r}}{q_0}\cdot\omega^2 \tag{VI.30}$$

which is consistent with VI.25 + VI.26.

The discussion about the sign in the particle picture together with VI.24 suggests to introduce some "gravitation charge" q_c with values $q_c=\{-1,0,1\}$. The charge-less, or neutral, case $q_c=0$ belongs to normal matter and anti-matter, while $q_c=-1$ for Њ-particles and $q_c=+1$ for anti-Њ-particles. We redefine

$$\omega\to q_c\cdot\omega,\ \ q\to q_c\cdot q\ \ \text{, where now}\ \ \omega, q\ \ \text{are positive functions and}\quad \omega_{,r}\ge 0,\ \ E_{,r}=H^3\omega_{,r}\ge 0\ . \tag{VI.31}$$

In almost all cases (equations) the sign of the charge cancels out, and so we have at most the values $q_c^2=1,0$. Hence we will furthermore not write this charge (or better only if needed).

Now we are able to analyze
-the influence of Њ-particles **on the scale factors** of the metric.
First we conclude for the Kepler Gravitation- "constant" $\gamma_k:=H^2=1+E^2$

$$\begin{aligned}&\text{i.)}\ \ \frac{\partial\gamma_k}{\partial r}\ge 0,\quad \frac{\partial\gamma_k}{\partial t}\le 0\quad\text{for}\quad \frac{\partial E^2}{\partial t}\le 0\ ,\\ &\text{ii.)}\ \ \frac{\partial\gamma_k}{\partial r}\le 0,\quad \frac{\partial\gamma_k}{\partial t}\ge 0\quad\text{for}\quad \frac{\partial E^2}{\partial t}\ge 0\end{aligned} \tag{VI.32}$$

In the first case γ_k decreases with time but increases with r .From this seems first immediately to follow, that the force acting on objects on circular orbits, increases with their radial distance from the center. But its need a bit more analysis. A growing γ_k also induces forces from the more far outside regions to the point of evaluation, as e.g. inside a ball of matter which increasing density in radial outgoing direction. A too fast growing γ_k would give a net effect of a decreasing force.
We calculate the acting force F in the Newton limit where (neglecting the factor 2)

$$F=-\frac{\partial r_h/r}{\partial r}=-\frac{\partial\gamma_k r_0/r}{\partial r}=-\gamma_k\frac{r_0}{r^2}\cdot\left(1-\gamma_k\cdot r\cdot(\omega^2)_{,r}\right)\ .$$

If we denote with $F_0:=-r_0/r^2,\ F_h:=-r_h/r^2$ the classical Newton forces with respect to the masses $r_{0,}r_h$, we arrive with $\gamma_k=1+\gamma_k\omega^2$ for the ratios

$$\begin{aligned} F/F_0 &= 1+\gamma_k\left(\omega^2 - r\cdot(\omega^2)_{,r}\right) \\ F/F_h &= 1 - r\cdot(\omega^2)_{,r} \end{aligned} \quad . \tag{VI.33}$$

So the net force is larger than the usual Newton force F_0 , if $\Delta = \omega^2 - r\cdot(\omega^2)_{,r} > 0$ and even larger as F_h if ω^2 is decreasing with r , that is in the second of the considered cases. As proofed above, for large r we have $(\omega^2)_{,r} \rightarrow (\omega_0^2)_{,r} \rightarrow 0$. If we use initial values that also $r(\omega_0^2)' \rightarrow 0$ for large r also $r(\omega^2)_{,r} \rightarrow 0$ holds and we receive

$$F/F_0 \rightarrow 1+\gamma_k\omega^2 = \gamma_k, \quad F/F_h = 1 \quad .$$

But for a radial decreasing $\omega \rightarrow 0$ and so a deceasing γ_k with limit 1 , this will only lead to the standard result $F \rightarrow F_h \rightarrow F_0$.
So we finally reject the case with a decreasing initial values ω_0^2 and with this not would get into the problem of velocities tending to infinite for large r .

As we will immediately see, initial values with the desired features (tending "fast" enough to a constant) are e.g. all functions which are analytic at infinity, i.e. which have a series expansion at infinity of the form $\omega_0(r) = \omega_\infty + \omega_1/r + O(1/r^2)$ or as another example $\omega_0 = \tanh(\lambda r), \lambda = const.$. In this last example and, if $\omega_\infty = 1$, also in those before, we will get in the limes $\gamma_k \rightarrow \infty$, which seems worrying. However for small enough parameter, even cosmic distances may be small on the scale ω_1/r or λr , respectively and VI.33 offers a possibility to explain the galaxy-rotation anomaly (called dark-matter effect). Further, applying the particle picture, the Ҍ-particles may not be equally distributed over the universe. Moreover we should not neglect, that the galaxies we observe are usually far away and so we measure their rotation as it have been a long time ago and that, a radial increasing γ_k decreases with time (see VI.33).

As next we consider the radial scale factor $-g_{rr} = Q^2/b^2$ for "empty" space $r_0 = 0$ (see VI.5 and VI.8). We will look again at its behavior as r tends to infinity. As we seen, for large r , all solutions, even the one for $r_0 \neq 0$, tend to the same solution (for the kind of initial values we consider) , and so the following results are valid for any r_0 . As said, we henceforth consider only a monotone increasing ω_0^2 , corresponding to VI.25 i), VI.26 I) and VI.32 I). We also use the replacements VI.31 such that we only have to deal with positive quantities $\omega = \omega_0 \geq 0$ and for the initial values $m = m_0 = q_c m_c = q_c\sqrt{(1-r^{-2})}$. The radial scale factor becomes to

$$\begin{aligned} -g_{rr} = Q^2 = f^2 + H^2 m^2 &= f^2 + \left(q_c H^3 \omega_{,r}\cdot t + q_c H m_c(r)\right)^2 \\ &= f^2 + q_c^2\left(E_{,r}\cdot t + H m_c(r)\right)^2 \\ &= H^2\left(1-\omega^2 f^2\right) + (E_{,r})^2\cdot t^2 + 2\mathrm{H}\, m_c(r) E_{,r} t \end{aligned} \quad ,$$

where now all summands are positive for $t \geq 0$. Therefor the time development,

$$\frac{-\partial g_{rr}}{\partial t} = 2\left((E_{,r})^2\cdot t + H\, m_c(r) E_{,r}\right) \quad . \tag{VI.34}$$

is strictly positive. This means, that the radial scale factor is growing with time as it does for an expanding universe. But this expansion is quite different to the typical one of a

Friedman-cosmos. In the last case, any point in the 3-D space expands uniformly in all directions due to

$$ds^2 = dt^2 - K(t)(dr^2 + r^2 d\Omega)$$, which looks like a growing ball.

In our case we have

$$ds^2 = dt^2 - (g^2(r,t)dr^2 + r^2 d\Omega) \quad ,$$

which means, that only the radial distance growth, but NOT the surface. A cut through the equator of the 3-D space will give a picture of tube or worm growing in the direction of an extra dimension. So if we measuring 3-D distances (at the same time), we measure between points with coordinates $(r_A, z(r_A,t))$, $(r_B, z(r_B,t))$.

For large r where $m_c \approx 1$ VI.34 is

$$\frac{-\partial g_{rr}}{\partial t} = 2(E_{,r}^2 \cdot t + H E_{,r}) \quad . \tag{VI.35}$$

We also consider the second derivatives

$$\frac{-\partial^2 g_{rr}}{\partial t^2} = 2 E_{,r}^2, \quad \frac{-\partial^2 g_{rr}}{\partial r \partial t} = 2 \frac{\partial}{\partial r}(E_{,r}^2 \cdot t + H E_{,r}) \quad . \tag{VI.36}$$

The first equations states, that the expansion rate increases with time (is accelerated), while the sign, which we will get of the mixed derivatives indicates if the expansion rate grows with distance. This sign depends also from the sign and value of second derivative of E . From our restrictions about the initial values we have $\lim_{r\to\infty} E_{,r} = \lim_{r\to\infty} H^3 \omega_{,r} = 0$. If $\omega_\infty := \lim_{r\to\infty} \omega$ and is smaller one, H is bounded and obviously also $\lim_{r\to\infty} HE_{,r} = 0$. But a limit value $\omega_\infty < 1$ would need some further explanation. On the other side for $\omega_\infty = 1 \Rightarrow \lim_{r\to\infty} H^2 = \infty$, as also $\lim_{r\to\infty} E = \infty$ and $\lim_{r\to\infty} g_{rr} = \infty$ holds. The question is again, like above for $\gamma_k \to \infty$, if this could be a realistic model. Beneath the arguments above, we can argue here, that due to VI.34, g_{rr} already tends to infinity for time goes to infinity. In such a way an infinite limit for growing r , even looks more symmetric. The values $\omega_\infty = 1,\ E_\infty = \infty$ mean, that even in the absence of any mass, i.e. $r_0 = 0$, the Ҍ-particles are "concentrated at infinity".
Lets consider the examples mentioned above. For an ω , analytic at infinity, we have $\omega = \omega_\infty - \omega_1/r + O(x^{-2})$. An example would be $\omega = \frac{2}{\pi} \operatorname{atan}(x)$ with $x = r/L$ or $x = (r-1)/L$ (if we want just handle $r \geq 1$) and $\omega_1 = \frac{2L}{\pi}$.
For this kind of initial values, we get in the limit up to terms of size $O(r^{-2})$

$$1 - \omega^2 = 2\omega_1/r \Rightarrow H^2 = r/(2\omega_1),\ \ HE_{,r} = H^4 \omega_{,r} = \frac{1}{4\omega_1},\ \ (E_{,r})^2 = (HE_{,r})^2/H^2 = \frac{1}{8\omega_1 r}$$

and so

$$\frac{-\partial g_{rr}}{\partial t} = \frac{1}{2\omega_1}\left(\frac{t}{2r} + 1\right) + O(r^{-2})$$. If we assume the light needs

approximately a time span of $t = r$ until we see it, we end up with

$$\frac{-\partial g_{rr}}{\partial t} = \frac{3}{4\omega_1} + O(r^{-2}) \quad .$$

In anyway, the expansion rate tends for large r to a constant, $\frac{-\partial^2 g_{rr}}{\partial r \partial t} = O(r^{-2})$.

In the second example from above, $\omega_0 = \tanh(\lambda r)$, we have $H = \cosh(\lambda r),\ E = \sinh(\lambda r),\ HE_{,r} = \lambda H^{2,},\ (E_{,r})^2 = \lambda^2 H^2$ and so

$$\frac{-\partial g_{rr}}{\partial t} = 2\lambda \cosh^2(\lambda r)(\lambda t + 1) \quad ,$$

which is infinity increasing in (positive) time and space direction.

At the end we consider also a

-simple, alternative approach, in which ω acts just as the mass generating field. For this, we have to start again with (VI.2) , but now using the flat space $a=1$ and $\dot{t}=1$, which results in (VI.5) with

$$ds^2 = b^2 dt^2 - \frac{Q^2}{b^2} dr^2,\ b^2 = 1 - \omega^{2,}\ Q^2 = b^2 f^2 + m^2,\ f = 1/r \quad . \tag{VI.37}$$

Now m must have again the form $m = \omega' t + m_c$. We may choose again m_c to be $m_c^2 = 1 - f^2$ or alternative $m_c^2 = 1 - b^2 f^2$. So $Q^2 = 1 - \omega^2 f^2 + 2\mathrm{m}_c \omega' t + (\omega' t)^2$ or just $Q^2 = 1 + 2\mathrm{m}_c \omega' t + (\omega' t)^2$ in the second case. Now, if we set $\omega = r_0/r$, we get the usual Schwarzschild factor $b^2 = 1 - r_0/r$ and for $r_0 = 0$ the usual flat space with $b^2 = Q^2 = 1$. In generally, an ω which dependents from r_0 , such it will become some constant for $r_0 = 0$ would make the behavior of ω more symmetric in time and space.

VII Discussion and Summary

In section V we discussed if there exists two different kind of matter, one which interacts with the field $d\beta$ and the other not. In the context of section VI we should reformulate this statement perhaps in the following way. The locally existence of the postulated particle or creation as particle anti-particle pair depends on the existence of some other physical quantity or field. If some kind of matter carry/interact with this quantity/field, even in very small areas, also our considered field will be present and consequently modifies the metric.

In section IV we found, that $d\beta$ can be seen also as an electromagnetic gauge-field. This in mind, we perhaps may assume that the existence of our field/particles is in some way related to electromagnetic fields, i.e. to photons and they will "see" are modified metric. So what we measure and call the speed of light is $1 = \mathbf{v} = Q\mathbf{v_0}$ and it will depend on the distance. In the neighborhood of particles of the second kind, no $d\beta$ is there and so the distance between two points at coordinate r_A and r_B is smaller and we would measure a velocity $\mathbf{v} = \mathbf{v_0}$, which is maybe faster than the one of photons. For example if $\mathbf{v_0}$ is the same in both cases we will get for particles of the second kind $\mathbf{v} = 1/Q > 1$. Candidates of particles of the second kind, should be those particles, which are not source of an em-field or its carrier. Without interaction boson, neutrinos are the only type of elementary particles with this characteristics and so they should be the first candidates for matter of the second kind. If this is true, this might could explain the effect measured by the OPERA project [OP] , if this effect really could be acknowledged.

After all I want to offer two further thoughts. The first arises from combining a minimal Schwarzschild radius of $min(r_0)=1/2$ and the minimal radial resolution 1 . So one we may wright for general Schwarzschild radii $r_0=(n+1/2)$, which is the energy spectrum of a quantum mechanical harmonic oscillator with unit energy.
For the second one, consider the AdS metric. As the Einstein-equation immediately shows (see Appendix A), **AdS** is a model for "something" with positive energy ($\Lambda>0$) and attractive forces. Usually **AdS** is used as cosmological model, but it also can seen as model of an ideal body, with constant density (and pressure). Consider the scale factor $g_{tt}=1-b(r)/r$ with $b(r)=r_0$ (Schwarzschild) and $b(r)/r=r^2/\alpha^2\equiv\Lambda/3\,r^2$ for **AdS** . In generally $b(r)$ is the mass inside the ball of radius r . So classically, with some mass density $\mu(r)$, one has $b(r)=\int_{x\leq r}\mu(r)\,dV$. If the mass density is constant and zero outside of some radius r_0 one has for $b(r)=4\pi/3\,\mu\,r_0^3 \text{ for } r\geq r_0$ and $b(r)=4\pi/3\,\mu\,r^3 \text{ for } r<r_0$. Comparing the second equation with **AdS** to identify Λ and inserting this into the first gives $b(r)=\Lambda/3\,r_0^3 \text{ for } r\geq r_0$. Now in this case $b(r)$ is the Schwarzschid radius and if it matches with the r_0 in the previous equation up to some constant factor one gets $\Lambda\,r_0^2=constant$. The scale factor may be written now as $g_{tt}=1-\lambda\cdot(r/r_0)^2$, which corresponds to a potential $V\propto(r/r_0)^2$. Such a potential is strong confined and asymptotically free for large energies r_0 .

A last observation I want to mention also. The results in III one may interpret roughly as "asymptotically anything is concentrated on a 2-D boundary", which is in some was similar to the "great attractor" - shell at r_f or also that for $\omega\rightarrow 1$ anything becomes hidden behind the horizon.

-Summarizing all

We offered an embedding of space-time in a 10 dimensional De-Sitter space, composed of an 8-D light cone and a Kaluza-Klein S^1 Sphere. The light cone itself consist of two copies of the 4-D space, with congruent spatial angles. The diameter of the KK-Sphere determines the curvature of the total space and defines a minimal space resolution for the embedding. From this it's deduced, that the diameter of the KK-dimension is of the size of a plank length. The minimal Schwarzschid radius, for which in the whole outer region the embedding exists is ½.

In section III , the space is considered in the context of basic complex projective geometry. This section is thought as proposal for scientist with a deeper knowledge in this subject to take a closer look on this kind of space for further researches to include quantum effects. The theory of Lagrangian submanifolds for example, may lead to Dirac like field equations as shown in [Ai] for C^2 . Ten dimensional spaces are also used in the Grand Unified Theory (the Georgi-Glashow model) and in some string theories. There is a large amount of literature (mathematical and physical) about the standard complex hyperbolic space (e.g. in AdS/CFT correspondence ...), physical literature concerning the complex de-Sitter space seems to be very sparsely (I found only [BEM]).

The subsequent section IV examines the proposed embedding in the context of classical Kaluza-Klein theory. Whereas the electromagnetic theory could not be inferred from this embedding, the set of geodesic submanifolds opens further possibilities.

One of them, which arises from a slightly extension of the phase of the Kaluza-Klein dimension, leads to a manifold with astonishing features (section V), depending from the time component ω of a simple field, looking like a standard gauge field.

– The space is asymptotically Ricci- flat.
– The vacuum energy density is negative.
– The event horizon r_h of any mass increases, corresponding to an extended Schwarzschild radius and so results in increased attractive force on circular orbits. This effect still exists in the Newton limit.
– On straight trajectories through the origin the Newton limit holds.
– Behind the horizon is a region, where the metric is euclidean.
– At some radial distance r_f , between the "naked" Schwarzschild radius r_0 and the extended r_h , exists a strong repulsive shell, like an impenetrable wall. Around this "wall" the matter is concentrated, that is the energy-momentum tensor fulfills the strong energy condition.

In the last section VI we seen, that the non-static extension leads to a theory, which allows us to assign a particle and its anti-particle to the field. In contrast to usual particles, for this one the particle have negative and the ant-particle positive energy. Further, the non static solutions offer possibilities to explain the "dark-matter effect" as well as the "dark-energy".

At all, the proposed embedding, seems to offer some new approaches for fundamental questions in physics.

Appendix A (Some properties of classical dS and AdS)

A general, isotropic, stationary metric is written in the form

$$ds^2 = a^2(r)\,dt^2 - b^{-2}(r)\,dr^2 - r^2\; d\,\Omega \tag{A1}$$

with $a^2(r) = 1 + U(r),\quad b^2(r) = 1 + V(r)$.

For $U = V > 0$ set $U = \sinh^2(\eta)$ and for $U = V < 0$ set $U = \text{-}\sin^2(\eta)$
the metric than is:

$$ds^2 = \cosh^2(\eta)\cdot dt^2 - d\,\eta^2 - r^2(\eta)\; d\,\Omega , \tag{A2}$$

resp. $$ds^2 = \cos^2(\eta)\cdot dt^2 - d\,\eta^2 - r^2(\eta)\; d\,\Omega$$

A Friedman-Lemaitre-Robertson-Walker (FLRW) has the form:

$$ds^2 = dt^2 - K^2(t)\cdot\left(d\,\eta^2 - r^2(\eta)\; d\,\Omega\right) \tag{A3}$$

Characteristic properties of **dS**:

– Hypersurface in $\boldsymbol{R}^{1,4}$: $x^2 \equiv \langle x , x \rangle \equiv x_0^2 - \sum_1^4 x_i^2 = -\alpha^2$

– Metric of $\boldsymbol{R}^{1,4}$: $d^2 s = \mathrm{d}\,x_0^2 - \sum_1^4 \mathrm{d}\,x_i^2$

– dS Metric (stationary): $U(r) = V(r) = -(r/\alpha)^2$
$r = \alpha\cdot\sin(\eta)$

– in FLRW coordinates $r = \sin(\eta),\, K = \alpha\cdot\cosh(t)$

– Newton potential: $U_N = \dfrac{U}{2} = \dfrac{-(r/\alpha)^2}{2}$

– Ricci curvature + Einstein tensor: $Ric = 3\cdot\alpha^{-2}\cdot g\,,\ \Rightarrow G = -Ric = -3\cdot\alpha^{-2}$
– Einstein equation: $G = -\Lambda\cdot g \Rightarrow \Lambda = 3\cdot\alpha^{-2}\cdot g > 0$

Characteristic properties of **AdS**:

– Hypersurface in $\boldsymbol{R}^{2,3}$: $x^2 \equiv \langle x , x \rangle \equiv x_0^2 + x_1^2 - \sum_2^4 x_i^2 = \alpha^2$

– Metric in $\boldsymbol{R}^{2,3}$: $\mathrm{d}^2 s = \mathrm{d}\,x_0^2 + \mathrm{d}\,x_1^2 - \sum_2^4 \mathrm{d}\,x_i^2$

– AdS Metric (stationary): $U(r) = V(r) = +(r/\alpha)^2$
$r = \alpha\cdot\sinh(\eta)$

– in FLRW coordinates $r=\sinh(\eta), K=\alpha\cdot\cos(t)$

– Newton potential: $U_N \quad =+\frac{1}{2}\cdot(\frac{r}{\alpha})^2$

– Ricci curvature + Einstein tensor : $Ric=-3\cdot\alpha^{-2}\cdot g\,, \quad \Rightarrow G=-Ric=3\cdot\alpha^{-2}$
– Einstein tensor : $G=-\Lambda\cdot g\Rightarrow\Lambda=-3\cdot\alpha^{-2}\cdot g<0$

Appendix B (Parametrization of dS and AdS in the 6 -D light cone)

In the following have a look how a global parametrization of **K**

$$\mathbf{K}=\{x\in\boldsymbol{R}^{2,4}:x^2:=\langle x\,,x\rangle=0\}$$

influences the induced metrics.

Using spherical coordinates in the last 3 space-like coordinates (which now again are treated as the usual space dimensions !), the metric of $\boldsymbol{R}^{2+4}$ reads as:

$$ds^2=dx_0^2+dx_1^2-dx_2^2-dr^2-r^2\,\mathrm{d}\,\Omega\,,$$

where $\mathrm{d}\,\Omega$ is the usual 3-d surface element and **K** is defined through the equation

$$x_0^2+x_1^2=x_2^2+r^{2.}\,. \qquad \text{(B1)}$$

1.) In circular parametrization,

$$x_0=a\cdot\sin(\omega),x_1=a\cdot\cos(\omega),x_2=a\cdot\cos(\phi),r=a\cdot\sin(\phi),\;\; a:=\cosh(\eta)$$

the metric on **K** becomes

$$ds^2=\cosh^2(\eta)\cdot(d\,\omega^2-d\,\phi^2-\sin^2(\phi)d\,\Omega)\quad . \qquad \text{(B2)}$$

Now **dS** is the section $x_1=\cosh(\eta)\cdot\cos(\omega)=1$. Putting this into the (B2) leads to the Friedman-Lemaitre-Robertson-Walker (FLRW) metric

$$ds^2=d\,\eta^2-\cosh^2(\eta)(d\,\phi^2+\sin^2(\phi)d\,\Omega)\,.$$

with η as the eigentime parameter and radial parameter $r=\sin(\phi)$ (note this parameter r now is not the same as the "original" one).

AdS is the section $x_2=\cosh(\eta)\cdot\cos(\phi)=1$. Applying this on (B2), induces the stationary Anti-de-Sitter metric

$$ds^2=\cosh^2(\eta)\cdot d\,\omega^2-d\,\eta^2-\sinh^2(\eta)d\,\Omega\quad .$$

with ω as the eigentime parameter and radial parameter $r=\sinh(\phi)$.

2.) In hyperbolic parametrization,

$$x_0=a\cdot\sinh(\omega),x_1=a\cdot\cosh(\phi),x_2=a\cdot\cosh(\omega),r=a\cdot\sinh(\phi),\;\; a:=\cos(\eta)$$

the metric on **K** is

$$ds^2 = \cos^2(\eta)\cdot(d\,\omega^2 - d\,\phi^2 - \sinh^2(\phi)\,d\,\Omega) \qquad \text{(B3)}$$

dS is the section $x_1 = \cos(\eta)\cdot\cosh(\phi) = 1$. Now, putting this into (B3), we get the stationary metric

$$ds^2 = \cos^2(\eta)\cdot d\,\omega^2 - d\,\eta^2 - \sin^2(\eta)\,d\,\Omega$$

with ω as the eigentime and radial parameter $r = \sin(\eta)$.

AdS is the section $x_2 = \cosh(\omega)\cdot\cos(\eta) = 1$. Again applying this relation to (B3), leads to the FLWR metric

$$ds^2 = d\,\eta^2 - \cos^2(\eta)(d\,\phi^2 + \sinh^2(\phi)\,d\,\Omega)$$

with η as the eigentime parameter and radial parameter $r = \sinh(\phi)$ (again this parameter r now is not the same as the "original" one).

Appendix C (8-D embedding)

On $R^{2,6}$ we start with the same parametrization as for $R^{2,8}$, but without the Kaluza-Klein sphere and not restrict the space on a sphere. The metric is equivalent to (II.2)

$$ds^2 = A^2\,d\,\omega^2 + dA^2 - dr^2 - r^2\,d\,\phi^2 - r^2\,d\,\Omega$$

but with no restriction for A. So set $\omega = t$ and $A = 1 + V(r)$. To get the metric of the form

$$ds^2 = A^2\,d\,\omega^2 - A^{-2}\,dr^2 - r^2\,d\,\Omega$$

we have to integrate just

$$r^2\,d\,\phi^2 = dA^2 - \frac{V}{1+V}\,dr^2 = (-V + (\frac{1}{2}\frac{dV}{dr})^2)\cdot\frac{dr^2}{1+V} \quad .$$

This is always possible, for the considered potentials $V = -r^2$ (**dS**), $V = +r^2$ (**AdS**) and $V = -r_0/r$ (**Sch).**

Appendix D (The metric of complex de-Sitter space)

In the following I point out the "motivation" for the projective metric in III.

Let $\langle .,. \rangle$ be the scalar product on $C^{d,n+1-d}$. On $M_h = \{z : \langle z, z \rangle = h\}$, $h \in \mathbb{R}$, define projective coordinates $u_j = \frac{z_j}{z_n}$, $j = 0 \ldots n$, with u_n = 1 (to keep this last constant coordinate is not usual but simplifies the notation in the following) .
The orthogonal component of a vector at z is an element of the kernel of the map du , that is vectors parallel to the coordinate "vector" z. The Cartesian coordinates $v:\ \{v_j,\ j = \ldots n\}$ of a vector in $C^{d,n+1-d}$ could be expressed through the coordinates $\hat{v}$ in the basis ∂_{u_i} (extended to n+1 dimensions) and the orthogonal vector on M_k

$$v = z_n \hat{v} + v_n u \text{ with } \hat{v}_n = 0 \qquad \text{(C1).}$$

We have also the usual decomposition of a vector in a tangential and orthogonal part $v = v^T + h^{-1} \cdot \langle v, z \rangle, \; z := v^T + b_v \cdot z$. Now define the induced metric on M to be just

$$(\hat{v}, \hat{w}) = \langle v^T, w^T \rangle \quad .$$

The left side is just an n-dimensional metric, the n-th components are zero (see C1)! Now inserting (C1) into $\langle v^T, w^T \rangle = \langle v, w \rangle - h \cdot b_v \bar{b}_w$ and using $|z_n|^2 \cdot \langle u, u \rangle = h$ and $\langle v, u \rangle = z_n \langle \hat{v}, u \rangle + v_n \langle u, u \rangle$ leads finally after some lines calculations to

$$(\hat{v}, \hat{w}) = |z_n|^2 (\langle \hat{v}, \hat{w} \rangle + |z_n|^2 \langle \hat{v}, u \rangle \langle u, \hat{w} \rangle) \quad .$$

Now we can drop the n-th component on the right side also. The final expression we obtains now $\langle u, u \rangle = \langle u, u \rangle_n + g_{nn}$, where $\langle u, u \rangle_n$ denotes also the reduced n-dimensional metric and $g_{nn} = \langle \partial_{z_n}, \partial_{z_n} \rangle$ is it's last diagonal entry. For $C^{1,4}$, $g_{nn} = -1$ and with h = -1 , $|z_n|^2 = \frac{1}{(1 - \langle u, u \rangle_n)}$, as in (III.1).

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